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Sulzbach, Henning; Neininger, Ralph; Drmota, Michael

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A Gaussian limit process for optimal FIND algorithms

Henning Sulzbach* Ralph Neininger† Michael Drmota‡

Abstract

We consider versions of the FIND algorithm where the pivot element used is the median of a subset chosen uniformly at random from the data. For the median selection we assume that subsamples of size asymptotic to $c \cdot n^\alpha$ are chosen, where $0 < \alpha \leq \frac{1}{2}$, $c > 0$ and n is the size of the data set to be split. We consider the complexity of FIND as a process in the rank to be selected and measured by the number of key comparisons required. After normalization we show weak convergence of the complexity to a centered Gaussian process as $n \rightarrow \infty$, which depends only on α . The proof relies on a contraction argument for probability distributions on càdlàg functions. We also identify the covariance function of the Gaussian limit process and discuss path and tail properties.

Keywords: FIND algorithm; Quickselect; complexity; key comparisons; functional limit theorem; contraction method; Gaussian process.

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1 Introduction

The FIND algorithm is a selection algorithm, also called Quickselect, to find an element of given rank ℓ in a set S of data, where the data set S is a subset of finite cardinality $|S|$ of some ordered set. We have $\ell \in \{1, 2, \dots, |S|\}$ and assume that the data are distinct. The algorithm was introduced by Hoare [22].

FIND is a one-sided version of the well-known sorting algorithm Quicksort. It works recursively by first choosing one element $p \in S$, called the pivot element, and generating two subsets $S_<$ and $S_>$, where $S_< := \{s \in S \mid s < p\}$ and $S_> := \{s \in S \mid s > p\}$. If $\ell = |S_<| + 1$ then the pivot element is the rank ℓ element to be selected and the algorithm stops. Otherwise, if $\ell \leq |S_<|$ it is recursively applied to $S_<$, if $\ell \geq |S_<| + 2$ it is recursively applied to $S_>$ searching for rank $\ell - |S_<| - 1$. This is called the 3-version

*School of Computer Science, McGill University, 3480 University Street, Montreal, Canada H3A 0E9.
 E-mail: henning.sulzbach@gmail.com

†Institute for Mathematics, J.W. Goethe University, 60054 Frankfurt am Main, Germany.
 E-mail: neininger@math.uni-frankfurt.de

‡Institute for Discrete Mathematics and Geometry, TU Vienna, A-1040 Vienna, Austria.
 E-mail: michael.drmota@tuwien.ac.at

of the algorithm, since the first partitioning step leads to three cases. A variant is the 2-version, where in the first partitioning step the sets $S_{\leq} := \{s \in S \mid s \leq p\}$ and $S_{>}$ are generated. Note that we have $p \in S_{\leq}$. We ignore the case where the pivot element is the rank ℓ element and recursively apply the algorithm to the subset among S_{\leq} and $S_{>}$ where the rank ℓ element is contained. Actually we will discuss both versions of the algorithm.

This specifies the FIND algorithm except for the choice of the pivot element in the partitioning step. It can be chosen as the first element of S , if S is given as a list (vector) so that a first element is well-defined, it can as well be chosen uniformly at random from S . In order to obtain better balanced subsets $S_{<}$ and $S_{>}$, respectively S_{\leq} and $S_{>}$, one may first choose a subset M of odd cardinality k from S and use the median of M as pivot element. This version is called the median-of- k FIND algorithm. Here k is fixed in advance and constant until the algorithm has performed all recursive calls and stops. A variant of FIND, which is discussed in the present paper, consists of letting $k = k(n)$ depend on n so that $1 \leq k(n) \leq n$ is odd and grows asymptotically as $c \cdot n^{\alpha}$ where $c > 0$, $0 < \alpha \leq \frac{1}{2}$ and $n \rightarrow \infty$. Note that in a recursive call on some $S' \subset S$ the subset of S' to choose the median from is of the size $k(|S'|)$. These routines turn out to be asymptotically optimal in a sense described below. First discussions of such versions can be found in [20] and more systematically in [35].

The algorithmic motivation for this version is to obtain even more balanced sublists. This results in algorithms which are efficient uniformly over the rank ℓ , hence they are reliable as universal algorithms to search for any rank $1 \leq \ell \leq n$. Note that one could also adapt the algorithm to select particular ranks ℓ . This is a different task; the literature is reviewed below.

For our subsequent probabilistic analysis we assume that the data are random variables in the unit interval $[0, 1]$, which are independent and identically distributed all with the uniform distribution on $[0, 1]$. Note that all our results also hold for any deterministic set of data as long as the subset to select the pivot element in each step is chosen independently and uniformly from the set of data. In our probabilistic model we also assume that the subset for the pivot selection is chosen independently from the data.

As a measure for the complexity we consider the number of key comparisons required by the version of FIND. We denote by $X_n^{(2)}(\ell)$ and $X_n^{(3)}(\ell)$ the number of key comparisons required when starting with a set of size n and selecting the element with rank $1 \leq \ell \leq n$ using the 2-version and 3-version respectively. Note that the choice of c and α as well as the particular choice of the median selection algorithm to find the pivot element within the subset are suppressed in the notation. A median of a set can be found in time (i.e. number of key comparisons) linear in the size of the set. It will later turn out that our results are independent of the choice of the median-selection algorithm to find the pivot element within the random subset as long as mild assumptions are satisfied which are shared by standard median-selection algorithms (we could in fact use FIND itself in this step). We denote the number of key comparisons needed to find the pivot as the median of a subset of size $k = k(n)$ by T_n and assume for any $p \geq 1$ that we have

$$\|T_n\|_p = O(k(n)), \quad (n \rightarrow \infty), \quad (1.1)$$

where $\|X\|_p := \mathbb{E}[|X|^p]^{1/p}$ denotes the L_p -norm of a random variable X for $1 \leq p < \infty$. The big-O notation as well as other Bachmann-Landau symbols are used here and later on.

The rank parameter ℓ is subsequently also interpreted as a time parameter of a stochastic process and we denote $X_n^{(2)} := (X_n^{(2)}(\ell))_{1 \leq \ell \leq n}$ and $X_n^{(3)} := (X_n^{(3)}(\ell))_{1 \leq \ell \leq n}$. In

Theorems 1.1 and 1.2 we state our main results about the asymptotic behavior of $X_n^{(2)}$ and $X_n^{(3)}$. Subsequently, we consider all appearing stochastic processes in time $t \in [0, 1]$ with càdlàg paths as random elements of the space $(\mathcal{D}[0, 1], d_{sk})$ of càdlàg functions on $[0, 1]$ with the Skorokhod metric d_{sk} , see Billingsley [3, Chapter 3].

Theorem 1.1. *Consider the process $X_n^{(2)} = (X_n^{(2)}(\ell))_{1 \leq \ell \leq n}$ of the number of key comparisons needed by the 2-version of the median-of- k FIND algorithm with $k = k(n) \sim cn^\alpha$ with $c > 0$ and $\alpha \in (0, \frac{1}{2}]$ and condition (1.1) for the pivot selection in the partitioning step. Then we have, as $n \rightarrow \infty$, the weak convergence*

$$\left(\frac{X_n^{(2)}(\lfloor tn \rfloor + 1) - 2n}{n^{1-\alpha/2}/\sqrt{c}} \right)_{t \in [0,1]} \xrightarrow{d} Z \quad \text{in } (\mathcal{D}[0, 1], d_{sk}),$$

where $Z = (Z_t)_{t \in [0,1]}$ is a centered Gaussian process depending on α with covariance function specified in Theorem 2.4 below (and where we set by convention $X_n^{(2)}(n+1) := X_n^{(2)}(n)$).

Our main convergence result for the 3-version is the weak convergence of all finite dimensional marginals, denoted by \xrightarrow{fdd} , for the analogously normalized process to the corresponding marginals of the Gaussian process of Theorem 1.1.

Theorem 1.2. *Consider the process $X_n^{(3)} = (X_n^{(3)}(\ell))_{1 \leq \ell \leq n}$ of the number of key comparisons needed by the 3-version of the median-of- k FIND algorithm with $k = k(n) \sim cn^\alpha$ with $c > 0$ and $\alpha \in (0, \frac{1}{2}]$ and condition (1.1) for the pivot selection in the partitioning step. Then we have, as $n \rightarrow \infty$, convergence of the finite dimensional marginals,*

$$\left(\frac{X_n^{(3)}(\lfloor tn \rfloor + 1) - 2n}{n^{1-\alpha/2}/\sqrt{c}} \right)_{t \in [0,1]} \xrightarrow{fdd} Z,$$

where $Z = (Z_t)_{t \in [0,1]}$ is the centered Gaussian process of Theorem 1.1 (and where we set by convention $X_n^{(3)}(n+1) := X_n^{(3)}(n)$).

Some additional related results are stated in Corollary 3.6.

As observed by Grübel [20], for the worst-case behavior of any version of FIND, we have

$$\liminf_{n \rightarrow \infty} \sup_{1 \leq \ell \leq n} \frac{X_n^{(3)}(\ell)}{n} \geq 2.$$

Moreover, Grübel [20, Theorem 5] notes that $\frac{1}{n} \sup_{1 \leq \ell \leq n} X_n^{(3)}(\ell) \rightarrow 2$ in probability for any median-of- k FIND variant with $k = k(n) \rightarrow \infty$ and $k = o(n/\log n)$. Hence, the algorithms investigated in the present work are asymptotically optimal with respect to the worst-case behavior. The following theorem gives more precise information.

Theorem 1.3. *As $n \rightarrow \infty$, with convergence of all moments, we have*

$$\frac{\sup_{1 \leq \ell \leq n} X_n^{(3)}(\ell) - 2n}{n^{1-\alpha/2}/\sqrt{c}} \rightarrow \sup_{t \in [0,1]} Z(t),$$

where $Z(t)$ is the process of Theorem 1.1. The same result holds for the 2-version.

In the classical case of FIND (by classical we mean with a uniformly chosen pivot element) a process convergence results for the number of key comparisons (as in Theorems 1.1 and 1.2) has been obtained in the seminal paper of Grübel and Rösler [21].

More precisely, if $X_n(\ell)$ denotes the number of key comparisons (in the 2-version) in classical FIND then

$$\left(\frac{X_n(\lfloor tn \rfloor + 1)}{n} \right)_{t \in [0,1]} \xrightarrow{d} (\tilde{Z}(t))_{t \in [0,1]} \quad \text{in } (\mathcal{D}[0,1], d_{sk}), \quad (1.2)$$

where $\tilde{Z} := (\tilde{Z}(t))_{t \in [0,1]}$ satisfies the stochastic fixed point equation

$$(\tilde{Z}(t))_{t \in [0,1]} \stackrel{d}{=} \left(1 + \mathbf{1}_{[0,U)}(t) U \tilde{Z}_0 \left(\frac{t}{U} \right) + \mathbf{1}_{[U,1]}(t) (1 - U) \tilde{Z}_1 \left(\frac{t - U}{1 - U} \right) \right)_{t \in [0,1]}.$$

Here, \tilde{Z}_0 and \tilde{Z}_1 have the same distribution as \tilde{Z} , U is uniformly distributed on $[0, 1]$, and $\tilde{Z}_0, \tilde{Z}_1, U$ are independent. In [21] also the difference between the 2-version and 3-version is discussed regarding weak convergence in $(\mathcal{D}[0, 1], d_{sk})$ for the 2-version, whereas for the 3-version such a convergence does not hold. A similar behavior appears for our FIND algorithm as reflected in Theorems 1.1 and 1.2.

For the classical FIND Paulsen [41] studied variances and higher moments in the setting of quantiles of [21]. Kodaj and Móri [31] investigated rates of convergence for the marginals of the process. Hwang and Tsai [23] considered the case $t = 0$, i.e. ranks of the form $\ell = o(n)$ and found (among other things) that here the limit distribution is the Dickman distribution. Note that this is the distribution of $\tilde{Z}(0)$.

With respect to the one-dimensional marginals, Theorem 1.1 and Theorem 1.2 reveal that, asymptotically, both first and second order behavior of the considered complexities do not depend on $t \in [0, 1]$. This stands in sharp contrast to the results for classical FIND (and median-of- k FIND with $k > 1$ fixed reviewed below), as the distribution of $\tilde{Z}(t)$ in (1.2) depends on t .

Historically the mathematical analysis of classical FIND was initiated with an average case analysis for fixed ranks ℓ by Knuth [30]. Variances were derived in Kirschenhofer and Prodinger [24].

For mathematical analysis of median-of- k versions of FIND with fixed k not depending on the size of the input we refer to Anderson and Brown [2], Kirschenhofer, Martínez and Prodinger [25] and Grübel [20]. A broad survey, also covering median-of- k analysis is given in Rösler [45].

A discussion of FIND versions with $k = k(n)$ depending on the size n of the list to be split with respect to the worst-case behavior was given in Grübel [20]. Martínez and Roura [35] give an average case analysis, where optimal choices for the tradeoff between better balanced sublists versus additional cost for the median selection are discussed. Note that another idea to adapt the FIND algorithm is to not choose the median of a subsample but to choose an element that may depend on the rank ℓ searched for such that the sublist where the algorithm is recursively called may be small. This is investigated in Martínez, Panario and Viola [36], see also Knof and Rösler [29, pp. 151–153].

In various contributions also the number of key exchanges is studied which has to be compared with the number of key comparisons for a more realistic measure of complexity. Corresponding limit distributions can be found in Hwang and Tsai [23], Knape and Neininger [26, Section 5], Mahmoud [32, 33] and Dadoun and Neininger [6].

Another model for the rank searched for is to consider a random rank chosen uniformly and independently from the data and algorithm. So called grand averages were considered for key comparisons in Mahmoud, Modarres and Smythe [34], and, for a different version of the partitioning stage using two pivot elements, in Wild, Nebel and Mahmoud [51]. For the number of key exchanges under grand averages see [33, 6]. Yet another complexity measure is the worst case complexity with worst case over the possible ranks, see Devroye [9].

Tail bounds for the number of key comparisons for the classical FIND were studied in Devroye [7] and Grübel [19].

A fundamentally different cost measure arises when a key comparison is weighted by the number of bit comparisons needed to identify its result. The number of bit comparisons was studied by Vallée et al. [50] and Fill and Nakama [16, 17], see also Grabner and Prodinger [18].

Finally we mention studies of exact simulation from distributions appearing as limit distributions in the analysis of FIND: Devroye [8], Fill and Huber [15], Devroye and Fawzi [10], Devroye and James [11] and [4, 27, 6].

The techniques used to show convergence in Section 3 and to construct the limit process Z in Section 2.1 are in the spirit of the contraction method. (We refer to Rösler and Rüschendorf [46] and Neininger and Rüschendorf [38] for an introduction and survey of the contraction method for univariate and finite-dimensional quantities.) In the last years a couple of general approaches have been developed to show process convergences within the contraction method on different function spaces and in different topologies, see Eickmeyer and Rüschendorf [14], Drmota, Janson and Neininger [12], Knof and Rösler [29], Neininger and Sulzbach [40] and Ragab and Rösler [43], as well as the PhD theses of Knof [28], Ragab [42] and Sulzbach [48].

The construction of the limit process Z that we present in Section 2.1 builds upon ideas of Ragab and Rösler [43]. However, the convergence proof for Theorem 1.1 yields weak convergence in $(\mathcal{D}[0, 1], d_{sk})$ which has to be compared with the convergence of finite dimensional distributions shown for a related problem in [43]. Our approach to convergence is almost entirely based on contraction arguments on the level of the supremum norm of processes and very little (deformation of time) is needed in addition to align jumps. Besides leading to comparatively strong results, we feel that the technique for convergence developed here is flexible and general to be easily applicable to related recursive problems.

A similar version of the Quicksort algorithm consists in also choosing the pivot element in each step as a median of a random sub-sample of size $k = k(n) \sim cn^\alpha$ with n the size of the list to be split. We conjecture that such a Quicksort algorithm admits a Gaussian limiting distribution for the normalized number of key comparisons. This would be in contrast to the well-known non-Gaussian limiting distribution for classical Quicksort, see [44].

Plan of the paper. The paper is organized as follows. In Section 2.1 the limit process Z is constructed and in Section 2.2 identified as a centered Gaussian process with explicitly given covariance function. Section 3 contains the asymptotic analysis of the complexity of the median-of- k FIND leading to the proofs of Theorems 1.1 and 1.2. The organization of the proofs is outlined at the beginning of Section 3. In the final Section 4 we present properties of the limit process Z . In Subsections 4.2 and 4.3 path properties of Z are discussed, Subsection 4.1 has a characterization and a tail bound for the supremum of the limit process Z . The Appendix is devoted to the proofs of two technical lemmata. The first, Lemma 5.1, allows the transfer of the results for the 2-version in Theorem 1.1 to the 3-version in Theorem 1.2. The second, Lemma 4.3, is needed in the study of the path variation of the limit process Z .

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2 Construction and characterization of the limit process

We first construct and characterize the limit process Z appearing in Theorems 1.1 and 1.2. In this and the following section we fix $\alpha \in (0, 1/2]$ and suppress the dependence on α in the notation.

2.1 Construction

We consider the rooted complete infinite binary tree, where the root is labeled by the empty word ϵ and left and right children of a node labeled ϑ are labeled by the extended words $\vartheta 0$ and $\vartheta 1$ respectively. The set of labels is denoted by $\Theta := \cup_{k=0}^{\infty} \{0, 1\}^k$. The length $|\vartheta|$ of a label of a node is identical to the depth of the node in the rooted complete infinite binary tree.

We denote the supremum norm on $\mathcal{D}[0, 1]$ by $\|\cdot\|$. For a random variable X in $(\mathcal{D}[0, 1], d_{sk})$ and $1 \leq p < \infty$ we denote the L_p -norm by $\|X\|_p := (\mathbb{E}[\|X\|^p])^{1/p}$.

For $u \in [0, 1]$ we define linear operators

$$\mathfrak{A}_u, \mathfrak{B}_u : \mathcal{D}[0, 1] \rightarrow \mathcal{D}[0, 1]$$

as follows. For $f \in \mathcal{D}[0, 1]$ the càdlàg functions $\mathfrak{A}_u(f)$ and $\mathfrak{B}_u(f)$ are defined as

$$t \mapsto \mathbf{1}_{\{t < u\}} f\left(\frac{t}{u}\right), \quad t \mapsto \mathbf{1}_{\{t \geq u\}} f\left(\frac{1-t}{1-u}\right),$$

respectively. Furthermore, we define the step function $\text{sg} : [0, 1] \rightarrow \mathbb{R}$ by $\text{sg}(t) = \mathbf{1}_{\{t < 1/2\}} - \mathbf{1}_{\{t \geq 1/2\}}$. Hence, sg is a shifted version the sign function, and it is in $\mathcal{D}[0, 1]$.

For a given family $\{N_\vartheta \mid \vartheta \in \Theta\}$ of independent random variables in \mathbb{R} each with the standard normal distribution we recursively define a family $\{Z_n^\vartheta \mid \vartheta \in \Theta, n \in \mathbb{N}_0\}$ of random variables in $(\mathcal{D}[0, 1], d_{sk})$ as follows: We set $Z_0^\vartheta := 0$ for all $\vartheta \in \Theta$. Assume, the Z_n^ϑ are already defined for an $n \geq 0$ and all $\vartheta \in \Theta$. Then for all $\vartheta \in \Theta$ we set

$$Z_{n+1}^\vartheta := \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{A}_{\frac{1}{2}}(Z_n^{\vartheta 0}) + \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{B}_{\frac{1}{2}}(Z_n^{\vartheta 1}) + N_\vartheta \cdot \text{sg}. \quad (2.1)$$

We have the following asymptotic properties for the Z_n^ϑ :

Lemma 2.1. *Let $\{Z_n^\vartheta \mid \vartheta \in \Theta, n \in \mathbb{N}_0\}$ be a family as defined (2.1). Then, for each $\vartheta \in \Theta$, the sequence $(Z_n^\vartheta)_{n \geq 0}$ converges almost surely uniformly and in the L_p -norm for all $p \in \mathbb{N}$ to a limit càdlàg process Z^ϑ . For all $\vartheta \in \Theta$ we have, almost surely,*

$$Z^\vartheta = \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{A}_{\frac{1}{2}}(Z^{\vartheta 0}) + \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{B}_{\frac{1}{2}}(Z^{\vartheta 1}) + N_\vartheta \cdot \text{sg}. \quad (2.2)$$

The family $\{Z^\vartheta \mid \vartheta \in \Theta\}$ is identically distributed and all moments of the $\|Z^\vartheta\|$ are finite.

Proof. We first show by induction that for all $\vartheta \in \Theta$ and all $n \in \mathbb{N}_0$ we have

$$\mathbb{E}[\|Z_{n+1}^\vartheta - Z_n^\vartheta\|^2] \leq 2^{-(1-\alpha)n}. \quad (2.3)$$

For $n = 0$ and $\vartheta \in \Theta$ we have $\mathbb{E}[\|Z_1^\vartheta - Z_0^\vartheta\|^2] = \mathbb{E}[\|N_\vartheta\|^2] = 1$, so (2.3) is satisfied for $n = 0$. Now as induction hypothesis assume, that (2.3) is true for all $\vartheta \in \Theta$ with n replaced by $n - 1$. Note that for a random variable X in $\mathcal{D}[0, 1]$ we have $\mathbb{E}[\|X\|^2] = \mathbb{E}[\|X^2\|]$ and that for all $f, g \in \mathcal{D}[0, 1]$ we have $\mathfrak{A}_u(f)\mathfrak{B}_u(g) = 0$ and $\|\mathfrak{A}_u(f)\| = \|\mathfrak{B}_u(f)\| = \|f\|$. With

these properties, (2.1) and the induction hypothesis we obtain

$$\begin{aligned} \mathbb{E} [\|Z_{n+1}^\vartheta - Z_n^\vartheta\|^2] &\leq \mathbb{E} \left[\left\| \left(\frac{1}{2} \right)^{1-\alpha/2} \mathfrak{A}_{\frac{1}{2}}(Z_n^{\vartheta 0} - Z_{n-1}^{\vartheta 0}) \right\|^2 + \left\| \left(\frac{1}{2} \right)^{1-\alpha/2} \mathfrak{B}_{\frac{1}{2}}(Z_n^{\vartheta 1} - Z_{n-1}^{\vartheta 1}) \right\|^2 \right] \\ &= \left(\frac{1}{2} \right)^{2-\alpha} \{ \mathbb{E} [\|Z_n^{\vartheta 0} - Z_{n-1}^{\vartheta 0}\|^2] + \mathbb{E} [\|Z_n^{\vartheta 1} - Z_{n-1}^{\vartheta 1}\|^2] \} \\ &\leq \left(\frac{1}{2} \right)^{2-\alpha} 2 \cdot 2^{-(1-\alpha)(n-1)} = 2^{-(1-\alpha)n}. \end{aligned} \quad (2.4)$$

From (2.3), using Markov's inequality, we infer that $\sup_{m \geq n} \|Z_m^\vartheta - Z_n^\vartheta\| \rightarrow 0$ as $n \rightarrow \infty$ in probability and hence $\sup_{m, p \geq n} \|Z_m^\vartheta - Z_p^\vartheta\| \rightarrow 0$ as $n \rightarrow \infty$ in probability by a simple application of the triangle inequality. By monotonicity, the latter convergence is almost sure. In other words, for each $\vartheta \in \Theta$, the sequence $(Z_n^\vartheta)_{n \geq 0}$ is almost surely a Cauchy sequence with respect to the $\|\cdot\|$ -norm. Since $(\mathcal{D}[0, 1], \|\cdot\|)$ is complete, there is a limit random process Z^ϑ such that we have convergence almost surely uniformly.

Since the operators $\mathfrak{A}_{\frac{1}{2}}$ and $\mathfrak{B}_{\frac{1}{2}}$ are continuous with respect to the $\|\cdot\|$ -norm we obtain (2.2) from (2.1) by letting $n \rightarrow \infty$. By construction, $\{Z_n^\vartheta \mid \vartheta \in \Theta\}$ is a family of identically distributed random variables for each $n \in \mathbb{N}_0$. Hence we obtain that the Z^ϑ are identically distributed. Finally, note that $Z_n^\vartheta = Z_0^\vartheta + \sum_{k=1}^n Z_k^\vartheta - Z_{k-1}^\vartheta$. Using (2.3) and the triangle inequality for the $\|\cdot\|_2$ -norm implies that $\mathbb{E} [\|Z_n^\vartheta\|^2]$ is bounded. The same arguments applied to the decomposition $Z^\vartheta = Z_0^\vartheta + \sum_{k=1}^\infty Z_k^\vartheta - Z_{k-1}^\vartheta$ show that $\mathbb{E} [\|Z_n^\vartheta - Z^\vartheta\|^2] \rightarrow 0$. Similar arguments apply for higher moments. \square

Definition 2.2. We write $Z := Z^\epsilon$, hence Z is a random process identically distributed as the Z^ϑ in Lemma 2.1 and call it the limit process and its distribution the limit distribution. Analogously we define $Z_n := Z_n^\epsilon$.

Let \mathcal{M} denote the set of probability measures on $(\mathcal{D}[0, 1], d_{sk})$. We define the map $T : \mathcal{M} \rightarrow \mathcal{M}$ by, for $\mu \in \mathcal{M}$,

$$T(\mu) := \mathcal{L} \left(\left(\frac{1}{2} \right)^{1-\alpha/2} \mathfrak{A}_{\frac{1}{2}}(X_0) + \left(\frac{1}{2} \right)^{1-\alpha/2} \mathfrak{B}_{\frac{1}{2}}(X_1) + N \cdot \text{sg} \right), \quad (2.5)$$

where $\mathcal{L}(X_0) = \mathcal{L}(X_1) = \mu$, N has the standard normal distribution and X_0, X_1, N are independent. For $1 \leq p < \infty$, we further denote

$$\mathcal{M}_p(\mathcal{D}[0, 1]) := \left\{ \mu \in \mathcal{M}(\mathcal{D}[0, 1]) \mid \int \|x\|^p d\mu(x) < \infty \right\}.$$

Let

$$p_\alpha = \frac{2}{2-\alpha}. \quad (2.6)$$

We have the following characterization of the limit distribution $\mathcal{L}(Z)$ of Z :

Lemma 2.3. Let $p > p_\alpha$. The limit distribution $\mathcal{L}(Z)$ from Definition 2.2 is the unique fixed-point of the restriction of T to $\mathcal{M}_p(\mathcal{D}[0, 1])$.

Proof. It is clear that $T(\mathcal{M}_p(\mathcal{D}[0, 1])) \subseteq \mathcal{M}_p(\mathcal{D}[0, 1])$. We endow $\mathcal{M}_p(\mathcal{D}[0, 1])$ with the following metric d : For $\mu, \nu \in \mathcal{M}_p(\mathcal{D}[0, 1])$ let

$$d(\mu, \nu) := \inf \left\{ (\mathbb{E} [\|X - Y\|^p])^{1/p} : \mathcal{L}(X) = \mu, \mathcal{L}(Y) = \nu \right\}.$$

To see that the restriction of T to $\mathcal{M}_p(\mathcal{D}[0, 1])$ is a strict contraction with respect to d let $\mu, \nu \in \mathcal{M}_p(\mathcal{D}[0, 1])$ be arbitrary, fix $\varepsilon > 0$ and choose random processes X and Y with $\mathcal{L}(X) = \mu$, $\mathcal{L}(Y) = \nu$ and $(\mathbb{E}[\|X - Y\|^p])^{1/p} \leq d(\mu, \nu) + \varepsilon$. Let (X', Y') be a copy of (X, Y) such that $N, (X, Y), (X', Y')$ are independent and N has the standard normal distribution. Then a calculation similar to (2.4) implies

$$\begin{aligned} d^p(T(\mu), T(\nu)) &\leq \left(\frac{1}{2}\right)^{p(1-\alpha/2)} (\mathbb{E}[\|X - Y\|^p] + \mathbb{E}[\|X' - Y'\|^p]) \\ &\leq 2^{1-p(1-\alpha/2)} (d(\mu, \nu) + \varepsilon)^p. \end{aligned}$$

With $\varepsilon \downarrow 0$ we obtain $d(T(\mu), T(\nu)) \leq 2^{1/p-(1-\alpha/2)} d(\mu, \nu)$. Hence, the restriction of T to $\mathcal{M}_p(\mathcal{D}[0, 1])$ is a strict contraction and has at most one fixed point. This implies the assertion. \square

2.2 Characterization of the limit process

For $\vartheta \in \Theta$ let B_ϑ be the set of real numbers in $[0, 1]$ whose binary representation has prefix ϑ . Here, the binary expansion of $t = t_1 t_2 \dots \in [0, 1]$ is uniquely determined by the convention that we always use expansions such that for all $k \in \mathbb{N}$ there exists $\ell > k$ with $t_\ell = 0$. Note that we have the decomposition $B_\vartheta = B_{\vartheta 0} \cup B_{\vartheta 1}$. The construction in (2.1) with the N_ϑ there implies representations for Z and Z_n from Definition 2.2, for all $t \in [0, 1]$ and $n \geq 0$:

$$\begin{aligned} Z_n(t) &= \sum_{\vartheta \in \Theta: |\vartheta| < n} \left(\frac{1}{2}\right)^{(1-\alpha/2) \cdot |\vartheta|} (\mathbf{1}_{\{t \in B_{\vartheta 0}\}} - \mathbf{1}_{\{t \in B_{\vartheta 1}\}}) N_\vartheta \\ Z(t) &= \sum_{\vartheta \in \Theta} \left(\frac{1}{2}\right)^{(1-\alpha/2) \cdot |\vartheta|} (\mathbf{1}_{\{t \in B_{\vartheta 0}\}} - \mathbf{1}_{\{t \in B_{\vartheta 1}\}}) N_\vartheta. \end{aligned} \quad (2.7)$$

Thus, Z_n is constant on the intervals $[i2^{-n}, (i+1)2^{-n})$ for $i = 0, \dots, 2^n - 1$. The $\vartheta \in \Theta$ with $|\vartheta| = n$ we denote in lexicographical order by $w_0, w_1, \dots, w_{2^n-1}$. Then we have

$$\begin{aligned} Z_{n+1}(t) - Z_n(t) &= \left(\frac{1}{2}\right)^{(1-\alpha/2) \cdot n} \sum_{i=0}^{2^n-1} (\mathbf{1}_{\{t \in B_{w_i 0}\}} - \mathbf{1}_{\{t \in B_{w_i 1}\}}) N_{w_i} \\ &= \left(\frac{1}{2}\right)^{(1-\alpha/2) \cdot n} \sum_{j=0}^{2^{n+1}-1} \mathbf{1}_{\{j2^{-(n+1)} \leq t < (j+1)2^{-(n+1)}\}} (-1)^j N_{w_{\lfloor j/2 \rfloor}}. \end{aligned}$$

For $u, v \in [0, 1]$ we denote their binary expansions by

$$u = \sum_{i=1}^{\infty} u_i 2^{-i}, \quad v = \sum_{i=1}^{\infty} v_i 2^{-i},$$

with $u_i, v_i \in \{0, 1\}$, again with the convention introduced above. Then we denote the length of the longest joint prefix of u and v in their binary expansions by

$$j(u, v) = \max\{j \geq 1 : (u_1, \dots, u_j) = (v_1, \dots, v_j)\},$$

with the conventions $\max \emptyset := 0$ and $\max \mathbb{N} := \infty$.

Theorem 2.4. *The limit process Z from Definition 2.2 is a centered Gaussian process with càdlàg paths. For its covariance function $\sigma(s, t) := \mathbb{E}[Z(t)Z(s)]$ we have*

$$\sigma(s, t) = \frac{\kappa^{j(s,t)+1} - 2\kappa^{j(s,t)} + 1}{1 - \kappa}, \quad \kappa := \left(\frac{1}{2}\right)^{2-\alpha} \quad (2.8)$$

with the convention $\kappa^\infty := 0$. Equivalently,

$$\mathbb{E}[(Z(t) - Z(s))^2] = \gamma \kappa^{j(s,t)}, \quad \gamma = \frac{4 - 2\kappa}{1 - \kappa}. \quad (2.9)$$

Proof. By induction we find that $(Z_n)_{n \geq 0}$ is a sequence of centered Gaussian processes. Hence, Lemma 2.1 implies that Z is a centered Gaussian process. It remains to compute the covariance function of Z . Comparing left and right hand side of equation (2.2) and using that, by construction, N_ϑ , $Z^{\vartheta 0}$ and $Z^{\vartheta 1}$ are independent, we find

$$\sigma(s, t) = \begin{cases} \kappa \sigma(2s, 2t) + 1, & \text{if } 0 \leq s, t < 1/2, \\ \kappa \sigma(2s - 1, 2t - 1) + 1, & \text{if } 1/2 \leq s, t \leq 1, \\ -1, & \text{if } 0 \leq s < 1/2 \leq t \leq 1. \end{cases}$$

From this it follows, for $s \neq t$ that

$$\sigma(s, t) = -\kappa^{j(s,t)} + \sum_{i=0}^{j(s,t)-1} \kappa^i.$$

By the theorem of dominated convergence, right-continuity of Z and the fact that $\mathbb{E}[\|Z\|^2] < \infty$ it follows, that for any $s \in [0, 1]$, $t \rightarrow \sigma(s, t)$ is right-continuous. This finishes the proof of (2.8). The equivalence with (2.9) is obvious. \square

For $k \in \mathbb{N}$ let $\mathcal{D}_k = \{i2^{-k} : i = 1, \dots, 2^k - 1\}$ and $\mathcal{D} = \bigcup_{k \geq 1} \mathcal{D}_k$ be the set of dyadic numbers on $(0, 1)$. For $t \in [0, 1)$ and a càdlàg function f , we define $f(t-) = \lim_{s \uparrow t} f(s)$ and $\Delta f(t) = f(t) - f(t-)$. Then, as Z is almost surely càdlàg, the previous theorem also implies

$$\mathcal{L}(\Delta Z(t)) = \mathcal{N}(0, \gamma \kappa^i) \quad (2.10)$$

for any $t \in \mathcal{D}$ where i is minimal with $t \in \mathcal{D}_i$. Here and subsequently, $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2 .

Corollary 2.5. *Almost surely, Z is continuous at t for all $t \notin \mathcal{D}$. On the contrary, for any $t \in \mathcal{D}$, almost surely, Z is not continuous at t .*

Proof. Let A be a set of measure one such that $Z_n \rightarrow Z$ uniformly on A . As Z_n is continuous at t for all n if $t \notin \mathcal{D}$ it follows that Z is continuous at t for all $t \notin \mathcal{D}$ on A , thus almost surely. For $t \in \mathcal{D}$, discontinuity follows immediately from (2.10). \square

More refined path properties are discussed in Sections 4.2 and 4.3. Simulations of realizations of Z_{10} for $\alpha = 1/2$ are presented below in Figure 1 to indicate the structure of the paths of the limit process Z .

3 Analysis of the Quickselect process

Our asymptotic analysis to prove the functional limit laws for the processes in Theorems 1.1 and 1.2 is organized as follows. In Section 3.1 we state a recurrence relation on which the whole analysis is based. To apply ideas from the contraction method we need to derive a distributional fixed point equation for a potential limit of the normalized processes as captured by the map T in (2.5). For this, in Section 3.2 first the asymptotic behavior of the size I_n of S_\leq is identified. Then in Section 3.3 a recurrence for the normalized processes appearing in Theorem 1.1 is derived. The random quantities are all embedded on one probability space and coupled in such a way that distances can be bounded pointwise (with respect to randomness ω) in the supremum norm on $\mathcal{D}[0, 1]$. We keep the jumps of a couple of auxiliary processes exactly aligned to those of Y_n in order to be able to bound distances by contraction arguments. The necessary deformations in time to align with the jumps of the limit process Z are afterwards done in Proposition 3.5.

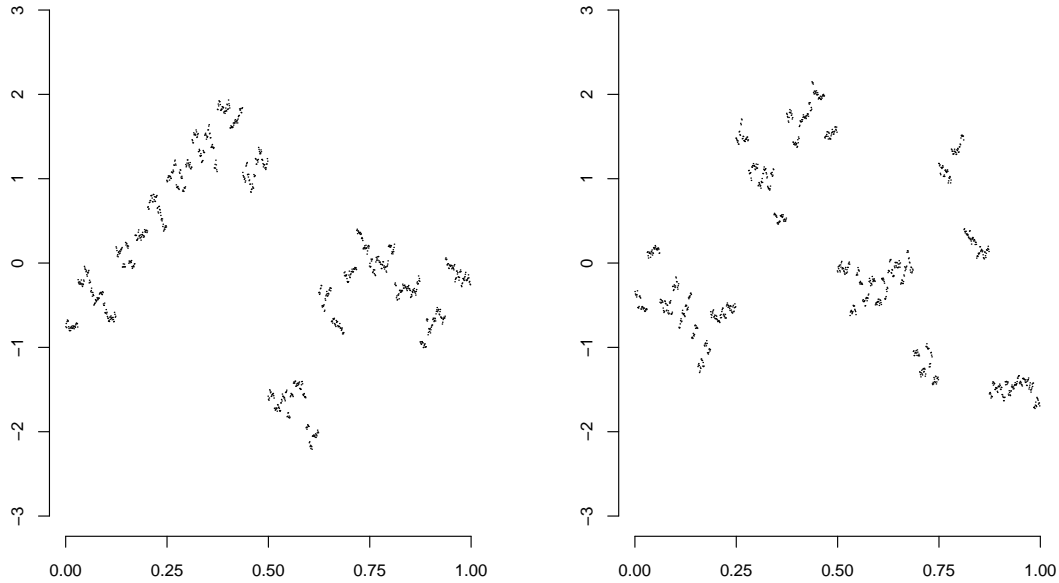


Figure 1: Simulations of two independent realizations of Z_{10} for $\alpha = 1/2$.

3.1 Preliminaries

Our analysis is based on a recurrence for the distributions of the processes $X_n^{(2)} = (X_n^{(2)}(\ell))_{1 \leq \ell \leq n}$ and $X_n^{(3)} = (X_n^{(3)}(\ell))_{1 \leq \ell \leq n}$. Note that after the selection of the median from the subset the k elements of the subset can already be assigned to the sets $S_<$, $S_>$ and $S_<$ respectively so that only $n - k$ remaining elements need to be compared with the pivot element. We denote the rank of the pivot element chosen in the first step by I_n . We set $X_0^{(2)} = X_0^{(3)} := 0$. Then we have $X_1^{(2)} = X_1^{(3)} = 0$ and, for all $n \geq 2$,

$$X_n^{(3)} \stackrel{d}{=} \left(\mathbf{1}_{\{\ell < I_n\}} X_{I_n-1}^{(3)}(\ell) + \mathbf{1}_{\{\ell \geq I_n+1\}} \widehat{X}_{n-I_n}^{(3)}(\ell - I_n) + n - k + T_n \right)_{1 \leq \ell \leq n}, \quad (3.1)$$

where $T_n, I_n, X_0^{(3)}, \dots, X_{n-1}^{(3)}, \widehat{X}_0^{(3)}, \dots, \widehat{X}_{n-1}^{(3)}$ are independent and $\widehat{X}_j^{(3)}$ is distributed as $X_j^{(3)}$ for $0 \leq j \leq n-1$. The stated independence is satisfied since in subsequent partitioning steps all choices of subsets are made independently. For the 2-version we have the same initial values as for the 3-version and, for all $n \geq 2$ that

$$X_n^{(2)} \stackrel{d}{=} \left(\mathbf{1}_{\{\ell < I_n+1\}} X_{I_n}^{(2)}(\ell) + \mathbf{1}_{\{\ell \geq I_n+1\}} \widehat{X}_{n-I_n}^{(2)}(\ell - I_n) + n - k + T_n \right)_{1 \leq \ell \leq n}, \quad (3.2)$$

with conditions on independence and identical distributions analogous to the 3-version in (3.1).

Recall that T_n is the number of key comparisons for the identification of the median within the random subset and that we assume condition (1.1).

We choose n_0 large enough such that $k(n) \geq 3$ for all $n \geq n_0$. This ensures that $I_n < n$ for all $n \geq n_0$.

3.2 Asymptotics for the pivot and sublist sizes

For simplicity of representation, we assume $c = 1$, i.e. $k = k(n) \sim n^\alpha$ with $\alpha \in (0, 1/2]$ for the remainder of the section. Elements in the presample of size k are chosen without

replacement, thus the distribution of I_n is given by

$$\mathbf{P}(I_n = i) = \frac{\binom{i-1}{(k-1)/2} \binom{n-i}{(k-1)/2}}{\binom{n}{k}}, \quad \frac{k+1}{2} \leq i \leq n - \frac{k-1}{2}. \quad (3.3)$$

Equivalently,

$$\mathcal{L}(I_n) = \mathcal{L}\left(\frac{k+1}{2} + \text{Bin}\left(n-k, \text{Beta}\left(\frac{k+1}{2}, \frac{k+1}{2}\right)\right)\right),$$

where, here and subsequently, for $n \in \mathbb{N}, p \in [0, 1]$, $\text{Bin}(n, p)$ denotes a random variable with the binomial distribution for n trials with success probability p . Moreover, for $\alpha, \beta > 0$, $\text{Beta}(\alpha, \beta)$ denotes a random variable with the beta distribution with parameters α, β .

Subsequently, let $(M_n)_{n \geq 1}$ be a sequence of random variables with the beta distribution with parameters $(k+1)/2, (k+1)/2$.

Lemma 3.1. *We have*

$$\mathbb{E}[M_n] = \frac{1}{2}, \quad \text{Var}(M_n) = \frac{1}{4(k+2)} \sim \frac{1}{4}n^{-\alpha},$$

and, for $n \rightarrow \infty$,

$$n^{\alpha/2} \left(M_n - \frac{1}{2}\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4}\right).$$

Proof. The expressions for mean and variance follow by straightforward calculations. For the limit theorem note that for the beta distribution and the binomial distribution we have the following identity

$$\mathbf{P}(\text{Beta}(a, b) < x) = \mathbf{P}(\text{Bin}(a+b-1, x) \geq a) \quad (3.4)$$

for all $a, b \in \mathbb{N}$ and $x \in (0, 1)$. Applying this to M_n and using the central limit theorem, e.g., in the version of de Moivre-Laplace implies the assertion. \square

For the size I_n of the left sublist generated in the first partitioning step we have:

Lemma 3.2. *We have*

$$\mathbb{E}[I_n] = \frac{n+1}{2}, \quad \text{Var}(I_n) = \frac{1}{4} \left(k(2n-k) + n \left(\frac{n-1}{k+2} + 1\right)\right) \sim \frac{1}{4}n^{2-\alpha}.$$

and

$$\frac{I_n - n/2}{n^{1-\alpha/2}} \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{4}\right).$$

Proof. The first two moments follow from Lemma 3.1. Given M_n , let X_n have the binomial distribution with parameters $n-k, M_n$ and set $I_n = \frac{k+1}{2} + X_n$. By Skorokhod's representation theorem, we may assume the existence of a sequence (\mathbf{F}_n) , where \mathbf{F}_n has the distribution of $n^{\alpha/2}(M_n - 1/2)$ such that $\mathbf{F}_n \rightarrow \mathbf{N}$ almost surely where \mathbf{N} has the normal $\mathcal{N}(0, \frac{1}{4})$ distribution. Let $\mathbf{M}_n = \mathbf{F}_n n^{-\alpha/2} + 1/2$ and construct \mathbf{X}_n and \mathbf{I}_n such as X_n and I_n but based on the \mathbf{M}_n . Decomposition yields

$$\begin{aligned} \frac{I_n - n/2}{n^{1-\alpha/2}} &= \frac{\mathbf{X}_n - \mathbf{M}_n(n-k)}{\sqrt{(n-k)\mathbf{M}_n(1-\mathbf{M}_n)}} \frac{\sqrt{(n-k)\mathbf{M}_n(1-\mathbf{M}_n)}}{n^{1-\alpha/2}} \\ &\quad + \frac{n\mathbf{M}_n - n/2}{n^{1-\alpha/2}} - \frac{k\mathbf{M}_n - k/2 + 1/2}{n^{1-\alpha/2}}. \end{aligned}$$

By construction, the second summand of the latter display tends to \mathbf{N} almost surely. Moreover, the third summand tends to zero almost surely. By conditioning on (\mathbf{M}_n) and the fact that $\mathbf{M}_n \rightarrow 1/2$ almost surely, the first factor of the first summand converges to a standard normal distribution by the central limit theorem for sums of independent and uniformly bounded random variables. As the second factor of the first summand tends to zero almost surely, the first summand converges to zero in probability. This shows

$$\frac{\mathbf{I}_n - n/2}{n^{1-\alpha/2}} \rightarrow \mathbf{N}, \quad (n \rightarrow \infty)$$

in probability. \square

More refined information about the distribution of M_n is given in the Appendix.

3.3 Proof of Theorems 1.1 and 1.2

We first discuss the 2-version of the process and recall the normalization from Theorem 1.1 which we denote by $Y_0 := 0$ and

$$Y_n(t) := \frac{X_n^{(2)}(\lfloor tn \rfloor + 1) - 2n}{n^{1-\alpha/2}}, \quad t \in [0, 1], n \geq 1,$$

with the convention $X_n^{(2)}(n+1) := X_n^{(2)}(n)$. Then, $Y_n := (Y_n(t))_{t \in [0,1]}$ satisfies, as a random variable in $(\mathcal{D}[0,1], d_{sk})$, for $n \geq n_0$ that

$$\begin{aligned} Y_n &\stackrel{d}{=} \left(\frac{I_n}{n} \right)^{1-\alpha/2} \mathfrak{A}_{\frac{I_n}{n}}(Y_{I_n}) + \left(\frac{n - I_n}{n} \right)^{1-\alpha/2} \mathfrak{B}_{\frac{I_n}{n}}(\hat{Y}_{n-I_n}) \\ &\quad + \frac{1}{n^{1-\alpha/2}} (T_n - k + \mathbf{1}_{\{t < I_n/n\}}(2I_n - n) + \mathbf{1}_{\{t \geq I_n/n\}}(n - 2I_n)) \end{aligned}$$

on $(\mathcal{D}[0,1], d_{sk})$ with conditions on independence and distributional copies as in (3.1).

Now, we embed all the relevant random variables on one probability space such that we have appropriate almost sure convergences. Throughout we use boldface characters to denote the embedded quantities. To be specific, by Skorokhod's representation theorem and Lemma 3.2, we can construct a set of independent and identically distributed random variates $\{(\mathbf{S}_n^\vartheta)_{n \geq n_0}, \mathbf{N}_\vartheta, \vartheta \in \Theta\}$ such that \mathbf{N}_ϑ has the standard normal distribution, \mathbf{S}_n^ϑ has the distribution of $(2I_n - n)/n^{1-\alpha/2}$ and $\mathbf{S}_n^\vartheta \rightarrow \mathbf{N}_\vartheta$ almost surely. Moreover, by Lemma 3.2, we have

$$\mathbb{E} [|\mathbf{S}_n^\vartheta - \mathbf{N}_\vartheta|^s] \rightarrow 0, \quad n \rightarrow \infty$$

for any $1 \leq s \leq 2$. Furthermore, note that $\mathcal{L}(I_n) = \mathcal{L}(\mathbf{J}_n^\vartheta)$ where $\mathbf{J}_n^\vartheta := \mathbf{S}_n^\vartheta \cdot n^{1-\alpha/2}/2 + n/2$. We can further augment this set of random variables by another set $\{\mathbf{T}_n^\vartheta, n \geq n_0, \vartheta \in \Theta\}$ of independent random variables, independent of $(\mathbf{S}_n^\vartheta)_{n \geq 0}, \mathbf{N}_\vartheta, \vartheta \in \Theta$, such that $\mathcal{L}(\mathbf{T}_n^\vartheta) = \mathcal{L}(T_n)$. Let $\mathbf{Y}_0^\vartheta := 0$ and $\{\mathbf{Y}_i^\vartheta \mid i < n_0, \vartheta \in \Theta\}$ be a set of independent processes with $\mathcal{L}(\mathbf{Y}_i^\vartheta) = \mathcal{L}(Y_i)$, also independent of the family of random variables defined above. For $n \geq n_0$, we define recursively

$$\begin{aligned} \mathbf{Y}_n^\vartheta &:= \left(\frac{\mathbf{J}_n^\vartheta}{n} \right)^{1-\alpha/2} \mathfrak{A}_{\frac{\mathbf{J}_n^\vartheta}{n}}(\mathbf{Y}_{\mathbf{J}_n^\vartheta}^{\vartheta 0}) + \left(\frac{n - \mathbf{J}_n^\vartheta}{n} \right)^{1-\alpha/2} \mathfrak{B}_{\frac{\mathbf{J}_n^\vartheta}{n}}(\mathbf{Y}_{n-\mathbf{J}_n^\vartheta}^{\vartheta 1}) \\ &\quad + \frac{1}{n^{1-\alpha/2}} (\mathbf{T}_n^\vartheta - k + \mathbf{1}_{\{t < \mathbf{J}_n^\vartheta/n\}}(2\mathbf{J}_n^\vartheta - n) + \mathbf{1}_{\{t \geq \mathbf{J}_n^\vartheta/n\}}(n - 2\mathbf{J}_n^\vartheta)). \end{aligned}$$

By construction, we have $\mathcal{L}(\mathbf{Y}_n^\vartheta) = \mathcal{L}(Y_n)$ for all $n \in \mathbb{N}$, since the sequences $(\mathbf{Y}_n^\vartheta)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ satisfy the same distributional recurrence and have the same initial distributions for $i = 0, \dots, n_0 - 1$. Subsequently, we use the sets $\{Z_n^\vartheta, n \in \mathbb{N}_0, \vartheta \in \Theta\}$ and

$\{Z^\vartheta, \vartheta \in \Theta\}$ as defined in (2.1) and Lemma 2.1 where the construction is executed using the particular set of random variables $\{N_\vartheta, \vartheta \in \Theta\}$. We denote the resulting random variables by $Z_n^\vartheta, n \in \mathbb{N}, \vartheta \in \Theta$ and $Z^\vartheta, \vartheta \in \Theta$.

To start bounding distances between Y_n and Z_n we use two intermediate sequences of stochastic processes Q_n^ϑ and R_n^ϑ in $(\mathcal{D}[0, 1], d_{sk})$. First, let $Q_i^\vartheta := 0$ for all $\vartheta \in \Theta, i < n_0$ and, recursively for all $n \geq n_0$,

$$Q_n^\vartheta := \left(\frac{J_n^\vartheta}{n}\right)^{1-\alpha/2} \mathfrak{A}_{\frac{J_n^\vartheta}{n}}(Q_{J_n^\vartheta}^{\vartheta 0}) + \left(\frac{n - J_n^\vartheta}{n}\right)^{1-\alpha/2} \mathfrak{B}_{\frac{J_n^\vartheta}{n}}(Q_{n-J_n^\vartheta}^{\vartheta 1}) \\ + \mathbf{1}_{\{t < J_n^\vartheta/n\}} N_\vartheta - \mathbf{1}_{\{t \geq J_n^\vartheta/n\}} N_\vartheta.$$

Second, $R_i^\vartheta := 0$ or all $\vartheta \in \Theta, i < n_0$ and, recursively for all $n \geq n_0$,

$$R_n^\vartheta := \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{A}_{\frac{J_n^\vartheta}{n}}(R_{J_n^\vartheta}^{\vartheta 0}) + \left(\frac{1}{2}\right)^{1-\alpha/2} \mathfrak{B}_{\frac{J_n^\vartheta}{n}}(R_{n-J_n^\vartheta}^{\vartheta 1}) \\ + \mathbf{1}_{\{t < J_n^\vartheta/n\}} N_\vartheta - \mathbf{1}_{\{t \geq J_n^\vartheta/n\}} N_\vartheta. \quad (3.5)$$

The proof of the functional limit law in Theorem 1.1 is organized by splitting the difference between Y_n^ϵ and Z^ϵ into several intermediate differences involving the terms defined above. As in Definition 2.2 we use the abbreviations $Y_n := Y_n^\epsilon, Q_n := Q_n^\epsilon, R_n := R_n^\epsilon$ and $Z_n := Z_n^\epsilon$.

Proposition 3.3. *As $n \rightarrow \infty$, we have $\mathbb{E}[\|Y_n - Q_n\|^2] \rightarrow 0$.*

Proposition 3.4. *As $n \rightarrow \infty$, we have $\mathbb{E}[\|Q_n - R_n\|^2] \rightarrow 0$.*

Proposition 3.5. *As $n \rightarrow \infty$, we have $d_{sk}(R_n, Z_n) \rightarrow 0$ in probability.*

These three propositions immediately yield $d_{sk}(Y_n, Z) \rightarrow 0$ in probability and thus Theorem 1.1. From this Theorem 1.2 follows from Theorem 1.1 and Lemma 5.1. The proof of Theorem 1.3 is given at the end of this section. Corollary 3.6 gives additional information. Here, for the sake of completeness, we formulate with a general the parameter $c > 0$ as in Theorem 1.1 and Theorem 1.2.

Corollary 3.6. *Let $t \notin \mathcal{D}$. If $t_n \rightarrow t$ then $Y_n(t_n) \rightarrow Z(t)$ in probability with convergence of all moments. Thus, for all $(\ell_n)_{n \geq 1}$ with $\ell_n \in \{1, \dots, n\}$ and $\ell_n/n \rightarrow t$ we have*

$$\frac{X_n^{(2)}(\ell_n) - 2n}{n^{1-\alpha/2}/\sqrt{c}} \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{1-\kappa}\right) \quad (n \rightarrow \infty)$$

together with convergence of all moments. The same is true for the 3-version $X_n^{(3)}$.

The rest of this section contains the proofs of our statements.

Proof of Proposition 3.3. By construction, we have

$$(Y_n(t) - Q_n(t))^2 \\ = \left(\frac{J_n}{n}\right)^{2-\alpha} \left(\mathfrak{A}_{\frac{J_n}{n}}(Y_{J_n}^{(0)} - Q_{J_n}^{(0)})\right)^2 + \left(\frac{n - J_n}{n}\right)^{2-\alpha} \left(\mathfrak{B}_{\frac{J_n}{n}}(Y_{n-J_n}^{(1)} - Q_{n-J_n}^{(1)})\right)^2 \\ + \frac{(T_n - k)^2}{n^{2-\alpha}} + \mathbf{1}_{\{t < J_n/n\}}(S_n - N)^2 + \mathbf{1}_{\{t \geq J_n/n\}}(S_n - N)^2 \quad (3.6)$$

$$+ 2 \frac{T_n - k}{n^{1-\alpha/2}} (\mathbf{1}_{\{t < J_n/n\}}(S_n - N) - \mathbf{1}_{\{t \geq J_n/n\}}(S_n - N)) \quad (3.7)$$

$$+ 2 \left(\frac{T_n - k}{n^{1-\alpha/2}} + \mathbf{1}_{\{t < J_n/n\}}(S_n - N) - \mathbf{1}_{\{t \geq J_n/n\}}(S_n - N) \right) \\ \times \left(\left(\frac{J_n}{n}\right)^{1-\alpha/2} \mathfrak{A}_{\frac{J_n}{n}}(Y_{J_n}^{(0)} - Q_{J_n}^{(0)}) + \left(\frac{n - J_n}{n}\right)^{1-\alpha/2} \mathfrak{B}_{\frac{J_n}{n}}(Y_{n-J_n}^{(1)} - Q_{n-J_n}^{(1)}) \right) \quad (3.8)$$

We now take the supremum over $t \in [0, 1]$ and the expectation on both sides. Then, by construction, the summands in lines (3.6) and (3.7) vanish as $n \rightarrow \infty$. Using the Cauchy-Schwarz inequality for the product in (3.8) and furthermore $\|\mathfrak{A}_u\| = \|\mathfrak{B}_u\| = 1$ we obtain altogether that

$$\begin{aligned} & \mathbb{E} [\|\mathbf{Y}_n - \mathbf{Q}_n\|^2] \\ & \leq \mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Y}_{\mathbf{J}_n}^{(0)} - \mathbf{Q}_{\mathbf{J}_n}^{(0)} \right\|^2 \right] + \mathbb{E} \left[\left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Y}_{n-\mathbf{J}_n}^{(1)} - \mathbf{Q}_{n-\mathbf{J}_n}^{(1)} \right\|^2 \right] \\ & \quad + \varepsilon_n \left(\mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Y}_{\mathbf{J}_n}^{(0)} - \mathbf{Q}_{\mathbf{J}_n}^{(0)} \right\|^2 \right] + \mathbb{E} \left[\left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Y}_{n-\mathbf{J}_n}^{(1)} - \mathbf{Q}_{n-\mathbf{J}_n}^{(1)} \right\|^2 \right] \right)^{1/2} \\ & \quad + \varepsilon'_n, \end{aligned} \quad (3.9)$$

where $\varepsilon_n, \varepsilon'_n \rightarrow 0$. Now, the arguments to infer $\mathbb{E} [\|\mathbf{Y}_n - \mathbf{Q}_n\|^2] \rightarrow 0$ are standard in the framework of the contraction method. In a first step, one shows that the sequence $\Delta_n := \mathbb{E} [\|\mathbf{Y}_n - \mathbf{Q}_n\|^2]$ is bounded. To this end, assume that $\Delta_m \leq C$ for all $m < n$ with $C \geq 1$. Then, the last display implies

$$\begin{aligned} \Delta_n & \leq C \left(\mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} + \left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \right] \right. \\ & \quad \left. + \varepsilon_n \left(\mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} + \left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \right] \right)^{1/2} \right) + \varepsilon'_n. \end{aligned}$$

As $\lim_{n \rightarrow \infty} \mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} + \left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \right] = 2^{-(1-\alpha)} < 1$, we can deduce $\Delta_n \leq C$ for all sufficiently large n . Then, one shows that $\limsup_{n \rightarrow \infty} \Delta_n = 0$ as follows. Start with denoting $D = \sup_{n \geq 0} \Delta_n$ and $\beta = \limsup_{n \rightarrow \infty} \Delta_n$. Let $\delta > 0$ be arbitrary and ℓ large enough such that $\Delta_n \leq \beta + \delta$ and $\mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} + \left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \right] \leq 2^{-(1-\alpha)} + \delta$ for all $n \geq \ell$. Moreover, we can assume n to be large enough to satisfy $\mathbf{P}(\ell \leq \mathbf{J}_n \leq n - \ell) \geq 1 - \delta$. Then, (3.9) implies

$$\Delta_n \leq D\delta + (\beta + \delta)(2^{-(1-\alpha)} + \delta) + \varepsilon_n \left(D\delta + (\beta + \delta)(2^{-(1-\alpha)} + \delta) \right)^{1/2} + \varepsilon'_n$$

Taking the limit superior on both sides and then letting $\delta \downarrow 0$ shows $\beta \leq 2^{-(1-\alpha)}\beta$. Thus, $\beta = 0$. \square

Proof of Proposition 3.4. By definition, we have

$$\begin{aligned} & \|\mathbf{Q}_n - \mathbf{R}_n\| \\ & \leq \left\| \left(\left(\frac{\mathbf{J}_n}{n} \right)^{1-\alpha/2} - \left(\frac{1}{2} \right)^{1-\alpha/2} \right) \mathfrak{A}_{\frac{\mathbf{J}_n}{n}} \left(\mathbf{R}_{\mathbf{J}_n}^{(0)} \right) \right. \\ & \quad \left. + \left(\left(\frac{n - \mathbf{J}_n}{n} \right)^{1-\alpha/2} - \left(\frac{1}{2} \right)^{1-\alpha/2} \right) \mathfrak{B}_{\frac{\mathbf{J}_n}{n}} \left(\mathbf{R}_{n-\mathbf{J}_n}^{(1)} \right) \right\| \\ & \quad + \left\| \left(\frac{\mathbf{J}_n}{n} \right)^{1-\alpha/2} \mathfrak{A}_{\frac{\mathbf{J}_n}{n}} \left(\mathbf{R}_{\mathbf{J}_n}^{(0)} - \mathbf{Q}_{\mathbf{J}_n}^{(0)} \right) + \left(\frac{n - \mathbf{J}_n}{n} \right)^{1-\alpha/2} \mathfrak{B}_{\frac{\mathbf{J}_n}{n}} \left(\mathbf{R}_{n-\mathbf{J}_n}^{(1)} - \mathbf{Q}_{n-\mathbf{J}_n}^{(1)} \right) \right\| \end{aligned}$$

Let ε''_n be the second moment of the first summand in the latter display. By construction, we have $\|\mathbf{R}_n\| \leq \|\mathbf{Z}_n\|$ for all $n \in \mathbb{N}$. Thus, Lemma 2.1 implies that the sequence

$\mathbb{E} [\|\mathbf{R}_n\|^2]$ is bounded. Using the Cauchy-Schwarz inequality, we infer that $\varepsilon_n'' \rightarrow 0$ as $n \rightarrow \infty$. Yet another application of the Cauchy-Schwarz inequality shows

$$\begin{aligned} & \mathbb{E} [\|\mathbf{Q}_n - \mathbf{R}_n\|^2] \\ & \leq \mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Q}_{\mathbf{J}_n}^{(0)} - \mathbf{R}_{\mathbf{J}_n}^{(0)} \right\|^2 \right] + \mathbb{E} \left[\left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Q}_{n-\mathbf{J}_n}^{(1)} - \mathbf{R}_{n-\mathbf{J}_n}^{(1)} \right\|^2 \right] \\ & \quad + \sqrt{\varepsilon_n''} \left(\mathbb{E} \left[\left(\frac{\mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Q}_{\mathbf{J}_n}^{(0)} - \mathbf{R}_{\mathbf{J}_n}^{(0)} \right\|^2 \right] + \mathbb{E} \left[\left(\frac{n - \mathbf{J}_n}{n} \right)^{2-\alpha} \left\| \mathbf{Q}_{n-\mathbf{J}_n}^{(1)} - \mathbf{R}_{n-\mathbf{J}_n}^{(1)} \right\|^2 \right] \right)^{1/2} \\ & \quad + \varepsilon_n''. \end{aligned}$$

The result now follows by an argument similar to the proof of Proposition 3.3. \square

Proof of Proposition 3.5. Let $\varepsilon > 0$. By Lemma 2.1 there exists an $n_1 \in \mathbb{N}$ such that

$$\mathbf{P} \left(\sup_{n \geq n_1} \|\mathbf{Z}_n - \mathbf{Z}_{n_1}\| > \varepsilon \right) \leq \varepsilon.$$

Let $n \geq n_1$. When applying the recurrence (3.5) for \mathbf{R}_n iteratively n_1 times we obtain a representation of \mathbf{R}_n with at most 2^{n_1} summands. Each summand corresponds to one of the 2^{n_1} sublists (some possibly being empty) generated by the algorithm in the first n_1 recursive steps. Let A_n denote the event that each of these 2^{n_1} sublists has size at least n_0 . On A_n the split into these first 2^{n_1} sublists causes $2^{n_1} - 1$ points of discontinuity of \mathbf{R}_n which we denote by $0 < T_n^1 < T_n^2 < \dots < T_n^{2^{n_1}-1}$. In fact, in general \mathbf{R}_n has additional points of discontinuity caused by splits when further unfolding the recurrence (3.5). Moreover, we denote the points of discontinuity of \mathbf{Z}_{n_1} by $\tau_n^k = k/2^{n_1}$ for $k = 1, \dots, 2^{n_1} - 1$.

By Lemma 3.2 we have $\mathbf{J}_n^\vartheta/n \rightarrow 1/2$ for each $\vartheta \in \Theta$ almost surely, hence

$$\mathbf{P} \left(A_n \cap \bigcap_{k=1}^{2^{n_1}-1} \{|T_n^k - \tau_n^k| < \varepsilon\} \right) \rightarrow 1, \quad (n \rightarrow \infty). \quad (3.10)$$

To bound the Skorokhod distance between \mathbf{R}_n and \mathbf{Z}_n we define a deformation of time as follows: On A_n let $\lambda_n : [0, 1] \rightarrow [0, 1]$ be defined by $\lambda_n(0) := 0, \lambda_n(1) := 1, \lambda_n(\tau_n^k) = T_n^k$ for $k = 1, \dots, 2^{n_1} - 1$ and linear in between these points. Then, with id the identity $t \mapsto t$ on $[0, 1]$ we have on the event in (3.10) that $\|\lambda_n - \text{id}\| < \varepsilon$. This implies for all $n \geq n_1$ that

$$\bigcap_{m \geq n_1} \{ \|\mathbf{Z}_m - \mathbf{Z}_{n_1}\| < \varepsilon \} \cap A_n \cap \bigcap_{k=1}^{2^{n_1}-1} \{|T_n^k - \tau_n^k| < \varepsilon\} \subseteq \{d_{sk}(\mathbf{R}_n, \mathbf{Z}_n) \leq 2\varepsilon\}.$$

To see this, note that on event on the left hand side, we have $\|\lambda_n - \text{id}\| \leq \varepsilon$ and

$$\|\mathbf{R}_n \circ \lambda_n - \mathbf{Z}_n\| \leq \|\mathbf{R}_n \circ \lambda_n - \mathbf{Z}_{n_1}\| + \|\mathbf{Z}_{n_1} - \mathbf{Z}_n\| \leq 2\varepsilon.$$

Thus, for all n sufficiently large, $\mathbf{P}(d_{sk}(\mathbf{R}_n, \mathbf{Z}_n) \leq 2\varepsilon) \geq 1 - 2\varepsilon$. \square

Proof of Corollary 3.6. Let $t \in [0, 1] \setminus \mathcal{D}$ and $(t_n)_{n \geq 1}$ a sequence in $[0, 1]$ with $t_n \rightarrow t$. By Proposition 3.3 we have $\mathbb{E} [\|\mathbf{Y}_n(t_n) - \mathbf{Q}_n(t_n)\|^2] \rightarrow 0$ as $n \rightarrow \infty$. Moreover, $d_{sk}(\mathbf{Q}_n, \mathbf{Z}) \rightarrow 0$ in probability by Propositions 3.4 and 3.5. As \mathbf{Z} is almost surely continuous at t , it follows that $\mathbf{Y}_n(t_n) \rightarrow \mathbf{Z}(t)$ in probability. Based on the uniform boundedness of the sequence $\mathbb{E} [\|\mathbf{Y}_n\|^2]$ a simple induction relying on its recursive definition shows that $\sup_{n \geq 1} \mathbb{E} [\|\mathbf{Y}_n\|^m] < \infty$ for all $m \in \mathbb{N}$. This implies the result for the 2-version. The statement about the 3-version follows from this and Lemma 5.1. \square

Proof of Theorem 1.3. Distributional convergence for the 2-version follows directly from Theorem 1.1. The proof of Theorem 1.1 has also revealed that $\|Z\|$ has finite moments of all orders and that the sequences $(\|Y_n\|)_{n \geq 1}$ and $(\|Q_n\|)_{n \geq 1}$ are both bounded in L_p for any $1 \leq p < \infty$. This shows the claim of Theorem 1.3 for the 2-version. An alternative approach which works for both the 2- and the 3-version relies on the contraction method for max-type recurrences. This is based on the distributional recurrence

$$W_n \stackrel{d}{=} \max(W_{I_n-1}, \widetilde{W}_{n-I_n}) + n - k + T_n,$$

where $W_n := \sup_{1 \leq \ell \leq n} X_n^{(3)}(\ell)$ and $(\widetilde{W}_n)_{n \geq 0}$ is an independent copy of $(W_n)_{n \geq 1}$, both independent of (I_n, T_n) . The latter display allows to deduce Theorem 1.3 straightforwardly from Theorem 4.6 in [47] together with the characterization of $\|Z\|$ given in Corollary 4.1. \square

4 Further properties of the limit process

In this section we first study the supremum of the limit process and derive tail bounds. Then path properties of the limit process Z are investigated. Here, first, the variation of the limit process Z is studied. Then, we will endow the unit interval with an alternative metric d_κ such that Z has continuous paths with respect to d_κ . This allows to study the modulus of continuity and Hölder continuity properties. In Sections 4.1 and 4.3, we make use of general results about path continuity and the supremum of Gaussian processes, see, e.g., Adler's book [1], and of the explicit construction of the limit process.

4.1 The supremum of the limit process

Let $S_n^\vartheta = \sup_{t \in [0,1]} Z_n^\vartheta(t)$ and $S^\vartheta = \sup_{t \in [0,1]} Z^\vartheta(t)$. By the uniform convergence stated in Lemma 2.1 we have $S_n^\vartheta \rightarrow S^\vartheta$ almost surely. The first result concerns a max-type recurrence for S_n and characterizes the distribution of S as solution of a stochastic fixed-point equation. To this end, let $\mathcal{M}(\mathbb{R})$ denote the set of probability measures on the real line,

$$\mathcal{M}_p(\mathbb{R}) := \left\{ \mu \in \mathcal{M}(\mathbb{R}) \mid \int |x|^p d\mu(x) < \infty \right\}, \quad 1 \leq p < \infty,$$

and $T^* : \mathcal{M}(\mathbb{R}) \rightarrow \mathcal{M}(\mathbb{R})$ be defined, for $\mu \in \mathcal{M}(\mathbb{R})$, by

$$T^*(\mu) := \mathcal{L} \left((\kappa^{1/2} X_0 + N) \vee (\kappa^{1/2} X_1 - N) \right),$$

where $\mathcal{L}(X_0) = \mathcal{L}(X_1) = \mu$, N has the standard normal distribution and X_0, X_1, N are independent, and $\kappa = 2^{\alpha-2}$ (as above).

Corollary 4.1. *Let $\vartheta \in \Theta$. We have*

$$\begin{aligned} S_{n+1}^\vartheta &= (\kappa^{1/2} S_n^{\vartheta 0} + N_\vartheta) \vee (\kappa^{1/2} S_n^{\vartheta 1} - N_\vartheta), \quad n \geq 1, \\ S^\vartheta &= (\kappa^{1/2} S^{\vartheta 0} + N_\vartheta) \vee (\kappa^{1/2} S^{\vartheta 1} - N_\vartheta) \quad \text{almost surely.} \end{aligned} \quad (4.1)$$

The distribution of S^ϑ is the unique fixed-point of the restriction of T^ to $\mathcal{M}_p(\mathbb{R})$ for any $p > p_\alpha$ with p_α given in (2.6).*

Proof. The recurrence for S_n^ϑ and the almost sure identity for S^ϑ follow by construction and Lemma 2.1. The characterization of $\mathcal{L}(S^\vartheta)$ is a special case of Theorem 3.4 in [39]. \square

It is a well-known phenomenon that the supremum of a Gaussian process resembles a Gaussian random variable. This explains the following proposition.

Proposition 4.2. *For the supremum $S = \sup_{t \in [0,1]} Z(t)$ of the limit process Z from Definition 2.2 we have for any $t > 0$ that*

$$\mathbf{P}(|S - \mathbb{E}[S]| \geq t) \leq 2 \exp\left(-\frac{1-\kappa}{2}t^2\right) \quad (4.2)$$

The same tail bounds are valid when S is replaced by $S_n = \sup_{t \in [0,1]} Z_n(t)$ for any $n \in \mathbb{N}$. The constant in the exponent on the right hand side of (4.2) is asymptotically optimal as $t \rightarrow \infty$. Moreover, we have

$$\frac{\sqrt{2}}{\sqrt{\pi}(1-\sqrt{\kappa})} \leq \mathbb{E}[S] \leq \sqrt{\frac{2}{1-2\kappa}}, \quad \text{Var}(S) \leq \frac{1}{1-\kappa}.$$

For $\alpha = 1/2$, the first bound leads to $\mathbb{E}[S] \in [1.968 \dots, 2.613 \dots]$.

Proof. From Theorem 2.4 we have $\text{Var}(Z(t)) = 1/(1-\kappa)$ for all $t \in [0, 1]$. The tail bound (4.2) now follows from a variant of Borell's inequality, see, e.g. Theorem 2.1 in [1]. For $t \rightarrow \infty$, optimality of the constant in the exponent follows directly by replacing S by $Z(t)$. The corresponding bound on $\text{Var}(S)$ can be deduced from Theorem 5.8 in [5] since there, the assumption of path continuity can be relaxed to regularity. Both results also apply to S_n for $n \in \mathbb{N}$.

For the lower bound on $\mathbb{E}[S]$ note that there is a $t_0 \in [0, 1]$ such that the terms $(\mathbf{1}_{\{t_0 \in B_{\vartheta 0}\}} - \mathbf{1}_{\{t_0 \in B_{\vartheta 1}\}})N_{\vartheta}$ in (2.7) are non-negative for all $\vartheta \in \Theta$. Hence, we obtain $\mathbb{E}[S] \geq \mathbb{E}[Z(t_0)] = \mathbb{E}[|N|] \sum_{i \geq 0} \kappa^{i/2}$, which is the lower bound.

For the upper bound on $\mathbb{E}[S]$ we take squares and expectations on left and right hand side of (4.1). This implies $\mathbb{E}[S^2] \leq 2/(1-2\kappa)$ and we obtain the bound from $\mathbb{E}[S] \leq \sqrt{\mathbb{E}[S^2]}$. \square

4.2 Variation of paths

We have already seen that the constant p_α defined in (2.6) is intimately linked to the limit process Z . In this section, we will see that this connection extends to path properties of Z , more precisely to its path variation. To formalize the main results of the section we need some notation. For $t \in (0, 1]$, let $\Pi(t)$ be the set of all finite decompositions of the interval $[0, t]$. Elements $\pi \in \Pi(t)$ we write as $\pi = \{\tau_1, \tau_2, \dots, \tau_k\}$ with $0 = \tau_1 < \tau_2 < \dots < \tau_k = t$. We also denote $|\pi| = k$ the size of π . Moreover, we abbreviate $\text{mesh}(\pi) = \max_{i=1, \dots, |\pi|-1} |\tau_{i+1} - \tau_i|$. For a càdlàg function f and $p > 0, t \in (0, 1]$, we define

$$V_{p,t}(f) := \sup_{\pi \in \Pi(t)} \sum_{i=1, \dots, |\pi|-1} |f(\tau_{i+1}) - f(\tau_i)|^p,$$

where $V_p(f) := V_{p,1}(f)$. Let N_f be the set of discontinuity points of f . Then, we set

$$W_{p,t}(f) := \sum_{s \in N_f \cap [0, t]} |\Delta f(s)|^p,$$

again with $W_p(f) := W_{p,1}(f)$. Finally, we set

$$[f]_t^{(p)} := \lim_{\substack{\pi \in \Pi(t) \\ \text{mesh}(\pi) \rightarrow 0}} \sum_{i=1, \dots, |\pi|-1} |f(\tau_{i+1}) - f(\tau_i)|^p, \quad (4.3)$$

if the limit exists in $\mathbb{R}_0^+ \cup \{\infty\}$. The càdlàg property of f implies that, for any $t \in (0, 1]$,

$$V_{p,t}(f) < \infty \Rightarrow W_{p,t}(f) < \infty \quad \text{and} \quad W_{p,t}(f) = \infty \Rightarrow [f]_t^{(p)} = \infty. \quad (4.4)$$

The following lemma is well-known in the case $p = 1, q = 2$, we did not find a proof for the general case in the literature. Thus, we include one in the Appendix.

Lemma 4.3. *Let $f \in \mathcal{D}[0, 1]$, $p > 0$ and $V_p(f) < \infty$. Then, for any $q > p$, we have*

$$[f]_t^{(q)} = W_{q,t}(f). \quad (4.5)$$

Additionally, the map $t \mapsto [f]_t^{(q)}$ is càdlàg with $\Delta[f]_t^{(q)} = |\Delta f(t)|^q$.

The following theorem is the main result of this section. Recall the definition of p_α in (2.6) and

$$\gamma = \frac{4 - 2\kappa}{1 - \kappa}. \quad (4.6)$$

Theorem 4.4. *i) For $p > p_\alpha$, we have that, almost surely, $V_p(Z) < \infty$ and*

$$[Z]_t^{(p)} = W_{p,t}(Z) = \sum_{s \in \mathcal{D} \cap [0, t]} |\Delta Z(s)|^p,$$

where the convergence in (4.3) with $f = Z$ also holds with respect to all moments. For the mean, we have

$$\mathbb{E} \left[[Z]_t^{(p)} \right] = \gamma^{p/2} \mathbb{E} [|N|^p] \sum_{i=1}^{\infty} \kappa^{pi/2} \left[2^{i-1} t + \frac{1}{2} \right].$$

ii) Almost surely, for any $t \in (0, 1]$, we have $V_{p_\alpha, t}(Z) = W_{p_\alpha, t}(Z) = [Z]_t^{(p_\alpha)} = \infty$.

The proof of the theorem makes use of a simple yet useful tool, well known, e.g., from Lévy's construction of Brownian motion.

Lemma 4.5. *Let $c > \sqrt{2 \log 2}$. Then, almost surely, there exists a (random) integer K such that for every $k \geq K$, we have*

$$\sup_{\vartheta \in \Theta: |v|=k} |N_v| \leq c \cdot k^{1/2}.$$

Proof. We have

$$\begin{aligned} & \sum_{k=1}^{\infty} \mathbf{P} \left(\sup_{\vartheta \in \Theta: |v|=k} |N_v| > ck^{1/2} \right) \\ & \leq \sum_{k=1}^{\infty} 2^{k+1} \mathbf{P} \left(N > ck^{1/2} \right) \leq \sum_{k=1}^{\infty} \frac{2^{k+1}}{\sqrt{2\pi}} \int_{ck^{1/2}}^{\infty} ye^{-y^2/2} dy = \sum_{k=1}^{\infty} \frac{2^{k+1}}{\sqrt{2\pi}} e^{-c^2 k/2} < \infty. \end{aligned}$$

The Borel–Cantelli Lemma implies the assertion. \square

Proof of Theorem 4.4. The main part of claim i) follows immediately from Lemma 4.3 upon establishing $V_p(Z) < \infty$ for $p > p_\alpha$ almost surely. To prove this, let A be a set of measure one and $K = K(\omega)$ for $\omega \in A$ such that the statement of Lemma 4.5 is satisfied with $c = 2$ there. Let $\pi \in \Pi(1)$. Then, for fixed $\omega \in A$,

$$\sum_{i=1}^{\pi-1} |Z(\tau_{i+1}) - Z(\tau_i)|^p = \sum_{\substack{i=1, \dots, \pi-1 \\ j(\tau_{i+1}, \tau_i) < K}} |Z(\tau_{i+1}) - Z(\tau_i)|^p + \sum_{\substack{i=1, \dots, \pi-1 \\ j(\tau_{i+1}, \tau_i) \geq K}} |Z(\tau_{i+1}) - Z(\tau_i)|^p \quad (4.7)$$

We will show that both terms on the right hand side can be bounded from above independently of the partition π . This shows the claim $V_p(Z) < \infty$. The first summand is easier. There are at most 2^ℓ pairs (τ_i, τ_{i+1}) such that $j(\tau_i, \tau_{i+1}) = \ell$. Thus,

$$\sum_{\substack{i=1, \dots, \pi-1 \\ j(\tau_{i+1}, \tau_i) < K}} |Z(\tau_{i+1}) - Z(\tau_i)|^p \leq 2^{K+1} \|Z\|^p. \quad (4.8)$$

Next, for $j(\tau_i, \tau_{i+1}) \geq K$,

$$|Z(\tau_{i+1}) - Z(\tau_i)|^p \leq \left(\sum_{\ell=j(\tau_i, \tau_{i+1})}^{\infty} 4\kappa^{\ell/2} \sqrt{\ell} \right)^p \leq (4D)^p \kappa^{j(\tau_{i+1}, \tau_i)p/2} j(\tau_{i+1}, \tau_i)^{p/2},$$

where we have abbreviated

$$D = \sum_{m=0}^{\infty} \kappa^{m/2} \sqrt{1+m}. \quad (4.9)$$

Summation implies

$$\sum_{\substack{i=1, \dots, \pi-1 \\ j(\tau_{i+1}, \tau_i) \geq K}} |Z(\tau_{i+1}) - Z(\tau_i)|^p \leq \sum_{j=K}^{\infty} 2^j (4D)^p \kappa^{jp/2} j^{p/2} \leq (4D)^p \sum_{j=K}^{\infty} (2\kappa^{p/2})^j j^{p/2}.$$

Since $2\kappa^{p/2} < 1$ the right hand side of the latter display is finite. Combining the latter display and (4.8), we obtain the desired upper bound for (4.7). For the convergence of moments let $m \in \mathbb{N}$. Then, for $\pi \in \Pi(1)$, we have

$$\left\| \sum_{i=0}^{|\pi|-1} |Z(\tau_{i+1}) - Z(\tau_i)|^p \right\|_m \leq \sum_{i=0}^{|\pi|-1} \| |Z(\tau_{i+1}) - Z(\tau_i)|^p \|_m \leq \gamma^{p/2} \| |N|^p \|_m \sum_{k=1}^{\infty} (2\kappa^{p/2})^k$$

The result follows as the last bound does not depend on π .

Regarding the mean of the p -variation, abbreviating $\mathcal{D}_0 = \emptyset$, we have

$$\begin{aligned} \mathbb{E} \left[[Z]_t^{(p)} \right] &= \sum_{s \in \mathcal{D} \cap [0, t]} \mathbb{E} [|\Delta Z(s)|^p] = \gamma^{p/2} \mathbb{E} [|N|^p] \sum_{i=1}^{\infty} |\mathcal{D}_i \setminus \mathcal{D}_{i-1} \cap [0, t]| \kappa^{pi/2} \\ &= \gamma^{p/2} \mathbb{E} [|N|^p] \sum_{i=1}^{\infty} \kappa^{pi/2} \left[2^{i-1} t + \frac{1}{2} \right] \end{aligned}$$

which finishes the proof of *i*).

We move on to the proof of *ii*). Due to (4.4) it is sufficient to show that, for any $t \in (0, 1]$, we have $W_{p_\alpha, t} = \infty$ almost surely. Again, we restrict our presentation to the case $t = 1$. As a warm-up we first investigate the case $p < p_\alpha$. Let $X_n = \sum_{t \in \mathcal{D}_n} |\Delta Z(t)|^p$ and $X'_n = \sum_{t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} |\Delta Z(t)|^p$. Then $X'_n \leq X_n \uparrow \sum_{t \in \mathcal{D}} |\Delta Z(t)|^p$ almost surely. The assertion $\sum_{t \in \mathcal{D}} |\Delta Z(t)|^p = \infty$ almost surely now follows easily from Chebychev's inequality and the facts that, as $n \rightarrow \infty$,

$$\begin{aligned} \mathbb{E} [X'_n] &= \sum_{t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} \mathbb{E} [|\Delta Z(t)|^p] = \frac{1}{2} \gamma^{p/2} \mathbb{E} [|N|^p] (2\kappa^{p/2})^n \rightarrow \infty, \\ \text{Var}(X'_n) &= \sum_{t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}} \text{Var}(|\Delta Z(t)|^p) = \frac{1}{2} \gamma^p \text{Var}(|N|^p) (2\kappa^p)^n = o \left(\mathbb{E} [X'_n]^2 \right). \end{aligned}$$

Here, we have used that the random variables $\Delta Z(t)$, $t \in \mathcal{D}_n \setminus \mathcal{D}_{n-1}$ are independent. Note that this does not extend to all $t \in \mathcal{D}_n$. The situation is more involved for $p = p_\alpha$. Here, the sequence $(\mathbb{E}[X'_n])$ is constant which implies

$$\mathbb{E}[X_n] = \frac{1}{2} \gamma^{1/(2-\alpha)} \mathbb{E}[|N|^{p_\alpha}] \cdot n.$$

Thus, $\mathbb{E}[[Z]_1^{(p_\alpha)}] = \infty$. The assertion now follows from showing that the variance of X_n grows at most linearly. By definition we have

$$\text{Var}(X_n) = \sum_{t \in \mathcal{D}_n} \text{Var}(|\Delta Z(t)|^{p_\alpha}) + \sum_{s, t \in \mathcal{D}_n, s \neq t} \text{Cov}(|\Delta Z(t)|^{p_\alpha}, |\Delta Z(s)|^{p_\alpha}).$$

First,

$$\sum_{t \in \mathcal{D}_n} \text{Var}(|\Delta Z(t)|^{p_\alpha}) \leq \gamma^{p_\alpha} \text{Var}(|N|^{p_\alpha}) \sum_{k=0}^{\infty} (2\kappa^{p_\alpha})^k,$$

where the right hand side does not depend on n . For $t \in \mathcal{D}_i \setminus \mathcal{D}_{i-1}$ and $j > i$, $\Delta Z(t)$ is independent of $\Delta Z(s)$ for all $s \in \mathcal{D}_j \setminus \mathcal{D}_{j-1}$ except for its direct neighbors. Thus, we have

$$\begin{aligned} & \sum_{s, t \in \mathcal{D}_n, s \neq t} \text{Cov}(|\Delta Z(t)|^{p_\alpha}, |\Delta Z(s)|^{p_\alpha}) \\ &= 2 \sum_{i=1}^n \sum_{t \in \mathcal{D}_i \setminus \mathcal{D}_{i-1}} \sum_{\substack{s \in \mathcal{D}_j \setminus \mathcal{D}_{j-1}, \\ i < j \leq n}} \text{Cov}(|\Delta Z(t)|^{p_\alpha}, |\Delta Z(s)|^{p_\alpha}) \\ &\leq 4 \sum_{i=1}^n \sum_{t \in \mathcal{D}_i \setminus \mathcal{D}_{i-1}} \sqrt{\text{Var}(|\Delta Z(t)|^{p_\alpha})} \sqrt{\gamma^{p_\alpha} \text{Var}(|N|^{p_\alpha})} \sum_{j>i} \kappa^{j/(3(1-\alpha))} \\ &\leq 4\gamma^{p_\alpha} \text{Var}(|N|^{p_\alpha}) \sum_{j=0}^{\infty} \kappa^{j/(3(1-\alpha))} \cdot n. \end{aligned}$$

The assertion follows. \square

4.3 Binary topology and path continuity

Regarding path continuity of a Gaussian process X on the unit interval, the canonical choice of a metric is given by $d(s, t) := \sqrt{\mathbb{E}[(X(t) - X(s))^2]}$ for $s, t \in [0, 1]$. In our case, that is $X = Z$, identifying $[0, 1]$ with $\{0, 1\}^{\mathbb{N}}$ via the binary representations, d induces the product topology on $\{0, 1\}^{\mathbb{N}}$. A sequence $(x^{(n)})_{n \geq 1}$ where $x^{(n)} = \sum_{i \geq 1} x_i^{(n)} 2^{-i}$ converges to x with respect to d if and only if for each $k \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that $x_i^{(n)} = x_i$ for all $i \leq k$ and $n \geq n_0$. Convergence $d(x^{(n)}, x) \rightarrow 0$ implies $|x^{(n)} - x| \rightarrow 0$. Conversely, $|x^{(n)} - x| \rightarrow 0$ implies $d(x^{(n)}, x) \rightarrow 0$ if and only if either $x \notin \mathcal{D}$ or $x \in \mathcal{D}$ and additionally $x^{(n)} \geq x$ for almost all n . The limit process Z as well as its p -variation for $p > p_\alpha$ are almost surely continuous with respect to d .

For notational reasons, we work with an (topologically) equivalent metric: for $x, y \in [0, 1]$ with binary representations $x = \sum_{i \geq 1} x_i 2^{-i}$, $y = \sum_{i \geq 1} y_i 2^{-i}$ we define

$$d_\kappa(x, y) := \kappa^{j(x, y)}. \quad (4.10)$$

Note again that d_κ and d depend on α via κ . Finally, working with d or more generally, changing the base in (4.10) to any value lower than one will only effect absolute constants in the following results.

The additive construction of Z somewhat resembles Lévy's construction of Brownian Motion which guides both intuition and proofs in the remainder of this section.

Theorem 4.6 (Modulus of continuity). *With γ as in (4.6) we have, almost surely,*

$$\sqrt{\frac{2\gamma \log 2}{\log(1/\kappa)}} \leq \limsup_{h \downarrow 0} \sup_{\substack{t, s \in [0, 1], \\ d_\kappa(t, s) = h}} \frac{|Z(t) - Z(s)|}{\sqrt{h \log(1/h)}} \leq \frac{2\sqrt{2 \log 2}}{\sqrt{\log(1/\kappa)(1 - \sqrt{\kappa})}},$$

where the \limsup is taken over sequences $h \downarrow 0$ with $h = \kappa^n$ for some $n \in \mathbb{N}$.

Proof. We start with the upper bound. First, let $\varepsilon > 0$ and K_1 be large enough such that $\sum_{j=0}^{\infty} \kappa^{j/2} \sqrt{1 + j/K_1} \leq (1 + \varepsilon)/(1 - \sqrt{\kappa})$. Next, let $c > \sqrt{2 \log 2}$ and choose $\omega \in A$ and $K \geq K_1$ as in Lemma 4.5. Let $h = \kappa^L$ with $L \geq K$. Then, for $t, s \in [0, 1]$ with $d_\kappa(t, s) = h$, it follows

$$|Z(t) - Z(s)| \leq 2c \sum_{j=L}^{\infty} \kappa^{j/2} \sqrt{j} \leq \frac{2c(1 + \varepsilon)}{\sqrt{\log(1/\kappa)(1 - \sqrt{\kappa})}} \sqrt{h \log(1/h)}.$$

Lower bounds follow analogously as for Brownian Motion.

Let $0 < v < \sqrt{2\gamma \log 2}$. For $n \in \mathbb{N}$ and $0 \leq k \leq 2^{n-1} - 1$ let

$$A_{k,n} = \{|Z((2k+1)2^{-n}) - Z(2k \cdot 2^{-n})| > v\sqrt{n}\kappa^{n/2}\}.$$

By construction, for fixed $n \in \mathbb{N}$, the family of events $A_{k,n}, 0 \leq k \leq 2^{n-1} - 1$ is independent. Moreover,

$$\mathbf{P}(A_{k,n}) = \mathbf{P}\left(|N| \geq \frac{v\sqrt{n}}{\sqrt{\gamma}}\right) \geq \Omega\left(\frac{1}{\sqrt{n}}\right) \exp\left(-\frac{v^2 n}{2\gamma}\right).$$

Thus, $2^{n-1} \mathbf{P}(A_{k,n}) \rightarrow \infty$ as $n \rightarrow \infty$. By independence,

$$\mathbf{P}\left(\bigcap_{k=0}^{2^{n-1}-1} A_{k,n}^c\right) \leq e^{-2^{n-1} \mathbf{P}(A_{1,n})} \rightarrow 0.$$

This yields the assertion upon choosing $h = \kappa^n$ for n sufficiently large (and random). \square

Moduli of continuity of the order $\sqrt{h \log(1/h)}$ can also be obtained from general results on Gaussian processes. First, by Theorem 4.6 in [1], which relies on deep results from Talagrand [49], a modulus of continuity is given by

$$\mathbb{E} \left[\sup_{d_\kappa(s,t) \leq h} (Z_s - Z_t) \right] = \sqrt{h} \mathbb{E} [\max\{S_1^*, \dots, S_{2^n}^*\}]$$

where $h = \kappa^n$ and $S_1^*, \dots, S_{2^n}^*$ are independent random variables, each having the distribution of $\sup_{s,t \in [0,1]} Z(t) - Z(s)$. An upper bound for the right hand side in the latter display by use of the bound (4.2) leads to a constant

$$\frac{4\sqrt{2 \log 2}}{\sqrt{1 - \kappa} \sqrt{\log 1/\kappa}},$$

which is slightly worse than the upper bound stated in Theorem 4.6. Second, the approach towards path continuity relying on the so-called metric entropy of $[0, 1]$ with respect to d_κ leads to a modulus of continuity of the same order with a random constant, see e.g. [13, Corollary 2.3].

Theorem 4.7 (Hölder continuity). *For any $\beta < 1/2$, almost surely, the paths of Z are Hölder continuous with exponent β with respect to d_κ . For any $\beta > 1/2$, almost surely, the paths of Z are nowhere pointwise Hölder continuous with exponent β with respect to d_κ .*

Proof. The result for $\beta < 1/2$ follows immediately from the upper bound on the modulus of continuity. Thus, we consider the case $\beta > 1/2$. We only treat the interval $[0, 1)$, the proof for $t = 1$ being easier. We adopt the proof of the corresponding statement for the Brownian Motion from [37], Theorem 1.30. As explained there, it is sufficient to show that, for any $M > 0$, the event

$$A = \left\{ \exists t \in [0, 1), \varepsilon > 0 : \sup_{s \in [t, 1], d_\kappa(t, s) < \varepsilon} |Z(s) - Z(t)| \leq M d_\kappa(t, s)^\beta \right\}$$

is a null event. We fix an integer $L > 4$ whose precise value will be specified later. For any $n > 3L$ let

$$R_n = \{0 \leq k \leq 2^n - 3L : d_\kappa((k + 3L)2^{-n}, k2^{-n}) \leq \kappa^{n-L}\}.$$

For $t \in [0, 1)$ and $n \in \mathbb{N}$, let $k_n(t) = \lfloor 2^n t \rfloor \in \mathbb{N}$ which satisfies $k_n(t)2^{-n} \leq t < (k_n(t) + 1)2^{-n}$. Then, with $t = (t_1, t_2, \dots)$, choose $m \in \mathbb{N}$ with $t_m = 0$ and set $n = n(m) = m + \lceil \log_2 3L \rceil$. (Note that there are infinitely many m with this property since $t \neq 1$.) Then, $d_\kappa((k_n(t) + 3L)2^{-n}, k_n(t)2^{-n}) \leq \kappa^{n - \lceil \log_2 3L \rceil}$. Hence, as $t \neq 1$, we have $k_n(t) \in R_n$ for infinitely many n . Moreover, for $k \in R_n$, we also have $d_\kappa((k + x)2^{-n}, k2^{-n}) \leq \kappa^{n-L}$ for $0 \leq x \leq 3L$ by monotonicity. Next, let

$$S_{n,k} = \{|Z((k + 3i)2^{-n}) - Z((k + 3i - 1)2^{-n})| \leq 2M\kappa^{(n-L)\beta} \forall 1 \leq i \leq L\}, \quad S_n = \bigcup_{k \in R_n} S_{n,k}.$$

Assume that $\omega \in A$ and $(t_0, \varepsilon_0) = (t_0(\omega), \varepsilon_0(\omega))$ satisfies the statement in the event A . Then, if $k_n(t_0) \in R_n$ and $n > 3L$ is large enough such that $\kappa^{n-L} < \varepsilon$, we infer

$$\begin{aligned} & |Z((k_n(t_0) + 3i)2^{-n}) - Z((k_n(t_0) + 3i - 1)2^{-n})| \\ & \leq |Z((k_n(t_0) + 3i)2^{-n}) - Z(t_0)| + |Z((k_n(t_0) + 3i - 1)2^{-n}) - Z(t_0)| \\ & \leq M(d_\kappa(t_0, (k_n(t_0) + 3i)2^{-n})^\beta + d_\kappa(t_0, (k_n(t_0) + 3i - 1)2^{-n})^\beta) \\ & \leq M(d_\kappa(k_n(t_0)2^{-n}, (k_n(t_0) + 3i)2^{-n})^\beta + d_\kappa(k_n(t_0)2^{-n}, (k_n(t_0) + 3i - 1)2^{-n})^\beta) \\ & \leq 2M\kappa^{(n-L)\beta} \end{aligned}$$

for all $1 \leq i \leq L$. Hence, $\omega \in S_{n,k_n(t_0)}$. As $k_n(t_0) \in R_n$ for infinitely many n , we can deduce that also $\omega \in S_n$ for infinitely many n , that is $A \subseteq \liminf S_n$. We finish the proof by showing that $\mathbf{P}(\liminf S_n) = 0$. For $k \in R_n$, we have

$$\begin{aligned} \mathbf{P}(S_{n,k}) &= \mathbf{P}\left(|Z((k + 3i)2^{-n}) - Z((k + 3i - 1)2^{-n})| \leq 2M\kappa^{(n-L)\beta} \forall 1 \leq i \leq L\right) \\ &= \prod_{i=1}^L \mathbf{P}\left(\gamma^{1/2}(d_\kappa((k + 3i)2^{-n}, (k + 3i - 1)2^{-n}))^{1/2}|N| \leq 2M\kappa^{(n-L)\beta}\right) \\ &\leq \left(\mathbf{P}\left(\gamma^{1/2}\kappa^{(n-L)/2}|N| \leq 2M\kappa^{(n-L)\beta}\right)\right)^L \end{aligned}$$

As the density of $|N|$ is bounded by 2, we have

$$\mathbf{P}(S_{n,k}) \leq (4M\kappa^{-L(\beta-1/2)}\gamma^{-1/2})^L \kappa^{nL(\beta-1/2)}.$$

Hence, as $|R_n| \leq 2^n$, by an application of the union bound, we see that the sequence $\mathbf{P}(S_n)$ is summable upon choosing $L > \max(4, 2/((2 - \alpha)(2\beta - 1)))$. Thus, $\mathbf{P}(\liminf S_n) = 0$ as desired. \square

5 Appendix

5.1 Refined information on the mean $\mathbb{E}[X_n^{(2)}(\ell)]$ and $\mathbb{E}[X_n^{(3)}(\ell)]$

We denote

$$c_n^{(2)}(\ell) := \mathbb{E}[X_n^{(2)}(\ell)], \quad c_n^{(3)}(\ell) := \mathbb{E}[X_n^{(3)}(\ell)], \quad n \geq 1, 1 \leq \ell \leq n. \quad (5.1)$$

The following result is sufficient to handle the difference between 2- and 3-version of the algorithm. Again, we assume $c = 1$.

Lemma 5.1. *For $c_n^{(2)}$ and $c_n^{(3)}$ defined in (5.1) we have $c_n^{(3)}(\ell) \leq c_n^{(2)}(\ell)$ for all $1 \leq \ell \leq n$ and, as $n \rightarrow \infty$,*

$$\sup_{1 \leq \ell \leq n} c_n^{(2)}(\ell) = O(n), \quad 0 \leq \sup_{1 \leq \ell \leq n} (c_n^{(2)}(\ell) - c_n^{(3)}(\ell)) = O\left(n^\alpha \sqrt{\log n}\right).$$

The proof of the lemma makes use of a tail bound for the distribution of M_n given in Lemma 5.2. It relies on standard concentration results for sums of independent random variables. The following simplified version of Bernstein's inequality, see e.g., Theorem 2.8 in [5] is sufficient: For a sequence of independent random variables X_1, \dots, X_n with $0 \leq X_i \leq 1$ for all $i = 1, \dots, n$, we have

$$\mathbf{P}\left(\left|\sum_{i=1}^n X_i - \mathbb{E}[X_i]\right| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2 \sum_{i=1}^n \mathbb{E}[X_i] + 2t/3}\right), \quad (5.2)$$

for all $t > 0$ and $n \in \mathbb{N}$.

Lemma 5.2. *Let $k \sim n^\alpha$ be odd. There exists a constant $C > 0$ such that for all $y > 0$ and $n \geq n_0$ we have*

$$\mathbf{P}\left(M_n - \frac{1}{2} > yn^{-\alpha/2}\right) \leq C \exp\left(-\frac{y^2}{4}\right).$$

Proof. Using the connection (3.4) with $x = 1/2 + yn^{-1/4}$ and $Y_n = \text{Bin}(k, x)$ we infer

$$\mathbf{P}\left(M_n - \frac{1}{2} > yn^{-\alpha/2}\right) \leq \mathbf{P}\left(Y_n - \mathbb{E}[Y_n] \leq \frac{1}{2} - kyn^{-\alpha/2}\right).$$

We may assume that $1 \leq y \leq n^{\alpha/2}$. Using Bernstein's inequality (5.2), we can deduce that for all n sufficiently large,

$$\mathbf{P}\left(M_n - \frac{1}{2} > yn^{-1/4}\right) \leq 2 \exp\left(-\frac{(kyn^{-\alpha/2} - \frac{1}{2})^2}{k + 8kyn^{-\alpha/2}/3 - 1/3}\right).$$

From here, the result follows easily. \square

Proof of Lemma 5.1. The claim $c_n^{(3)}(\ell) \leq c_n^{(2)}(\ell)$ is clear. Note that (3.2) implies

$$c_n^{(2)}(\ell) = \sum_{i=\ell}^n \mathbf{P}(I_n = i) c_i^{(2)}(\ell) + \sum_{i=1}^{\ell-1} \mathbf{P}(I_n = i) c_{n-i}^{(2)}(\ell - i) + n - k + \mathbb{E}[T_n] \quad (5.3)$$

for all $n \geq n_0$. Assuming that $c_i^{(2)}(\ell) \leq Ci$ for all $1 \leq \ell \leq i$ and $i \leq n-1$, it follows that

$$c_n^{(2)}(\ell) \leq C\mathbb{E}[\max(I_n, n - I_n)] + n - k + \mathbb{E}[T_n].$$

Choosing n and C large enough, the right hand side is bounded by Cn as $\mathbb{E}[T_n] = O(n^\alpha)$ which proves $\sup_{1 \leq \ell \leq n} c_n^{(2)}(\ell) = O(n)$.

Much in the same way as (5.3) follows from (3.2), the following recurrence follows from (3.1):

$$c_n^{(3)}(\ell) = \sum_{i=\ell+1}^n \mathbf{P}(I_n = i) c_{i-1}^{(3)}(\ell) + \sum_{i=1}^{\ell-1} \mathbf{P}(I_n = i) c_{n-i}^{(3)}(\ell - i) + n - k + \mathbb{E}[T_n],$$

for $n \geq n_0$. We proceed recursively and assume that $|c_i^{(2)}(\ell) - c_i^{(3)}(\ell)| \leq Ci^\alpha \sqrt{\log i}$ for all $1 \leq \ell \leq i$, $i \leq n-1$. Then, denoting $I_n^* = \max(I_n, n - I_n)$,

$$\begin{aligned} |c_n^{(2)}(\ell) - c_n^{(3)}(\ell)| &\leq \sum_{i=\ell}^{n-1} |\mathbf{P}(I_n = i+1) - \mathbf{P}(I_n = i)| c_i^{(3)}(\ell) \\ &\quad + \mathbf{P}(I_n = n) c_n^{(2)}(\ell) + C\mathbb{E}[(I_n^*)^\alpha \sqrt{\log I_n^*}] \\ &= \sum_{i=\ell}^{n-1} |\mathbf{P}(I_n = i+1) - \mathbf{P}(I_n = i)| c_i^{(3)}(\ell) + C\mathbb{E}[(I_n^*)^\alpha \sqrt{\log I_n^*}] \end{aligned} \quad (5.4)$$

For now, let us assume (a proof given below) that

$$\mathbf{P}(I_n = i+1) - \mathbf{P}(I_n = i) = O(n^{\alpha-2} \sqrt{\log n}) \quad (5.5)$$

uniformly in $1 \leq i \leq n$. By the results obtained so far, we have

$$c_n^{(3)}(\ell) \leq c_n^{(2)}(\ell) = O(n)$$

uniformly in $1 \leq \ell \leq n$. Using these two bounds, it follows from (5.4) that

$$|c_n^{(2)}(\ell) - c_n^{(3)}(\ell)| \leq (C^* n^\alpha + C\mathbb{E}[(I_n^*)^\alpha]) \sqrt{\log n}$$

for some universal constant $C^* > 0$. Let $\varepsilon > 0$ be sufficiently small and assume n was chosen large enough such that $\mathbb{E}[(I_m^*/m)^\alpha] \leq 1 - \varepsilon$ for all $m \geq n$. Then $|c_n^{(2)}(\ell) - c_n^{(3)}(\ell)| \leq Cn^\alpha \sqrt{\log n}$ follows upon choosing $C \geq C^*/\varepsilon$.

It remains to prove (5.5): First, observe that we can write

$$\mathbf{P}(I_n = i+1) - \mathbf{P}(I_n = i) = \mathbf{P}(I_n = i) \left(\frac{(k-1)(n-2i)}{(n-i)(2i-(k-1))} \right). \quad (5.6)$$

By symmetry, it is enough to consider the case $i \leq \lfloor n/2 \rfloor$. Moreover, again by symmetry, $\mathbf{P}(I_n = i)$ is maximal for $i = \lfloor n/2 \rfloor$. An application of Stirling's formula in (3.3) shows that

$$\sup_{1 \leq i \leq n} \mathbf{P}(I_n = i) = \mathbf{P}(I_n = \lfloor n/2 \rfloor) \sim \sqrt{\frac{2}{\pi}} n^{\alpha/2-1}.$$

In particular, $\sup_{1 \leq i \leq n} \mathbf{P}(I_n = i) = O(n^{\alpha/2-1})$, which is suggested by the limit law in Lemma 3.2. Now, let $C_1 > 0$ (to be specified later) and $\gamma_n = n/2 - C_1 \sqrt{\log n} n^{1-\alpha/2}$. For $i \geq \gamma_n$ it is easy to see that the second factor on the right hand side of (5.6) is uniformly bounded by a constant multiple of $\sqrt{\log n} n^{\alpha/2-1}$. Thus,

$$\sup_{\gamma_n \leq i \leq \lfloor n/2 \rfloor} \mathbf{P}(I_n = i+1) - \mathbf{P}(I_n = i) = O(n^{\alpha-2} \sqrt{\log n}).$$

To treat small values of i , consider the event $A_n := \{M_n \geq \frac{1}{2} - C_2 \sqrt{\log nn^{-\alpha/2}}\}$ with $0 < C_2 < C_1$. We have

$$\begin{aligned} \mathbf{P}(I_n < \gamma_n, A_n) &\leq \mathbf{P}\left(\text{Bin}(n-k, 1/2 - C_2 \sqrt{\log nn^{-\alpha/2}}) < \gamma_n - (k+1)/2\right) \\ &\leq \mathbf{P}\left(|Y_n - \mathbb{E}[Y_n]| \geq (C_1 - C_2) \sqrt{\log nn^{1-\alpha/2}}\right) \end{aligned}$$

where $Y_n = \text{Bin}(n-k, 1/2 - C_2 \sqrt{\log nn^{-\alpha/2}})$ and n is assumed to be sufficiently large. Using Bernstein's inequality (5.2), the latter display is bounded by a multiple of $\exp(-(C_1 - C_2)^2 n^{1-\alpha} \log n/2)$. By Lemma 5.2, $\mathbf{P}(A_n^c) = O(n^{-C_2^2/4})$ which finally shows that

$$\sup_{1 \leq i \leq \gamma_n} \mathbf{P}(I_n = i) = O\left(\max\left\{n^{-C_2^2/4}, \exp\left(-\frac{1}{2}(C_1 - C_2)^2 n^{1-\alpha} \log n\right)\right\}\right).$$

This finishes the proof of the lemma by choosing, e.g., $C_2 = 3$ and $C_1 = 4$. \square

5.2 Proof of Lemma 4.3

By (4.4) we have $\sum_{t \in N_f} |\Delta f(t)|^p =: M < \infty$. Let $\delta, \varepsilon > 0$. There exists a number $N \in \mathbb{N}$ and a set $N'_f = \{\sigma_i : 1 \leq i \leq N\} \subseteq N_f$ with $0 < \sigma_1 < \dots < \sigma_N \leq 1$ such that, first,

$$\begin{aligned} |f(t) - f(s)| &\leq \delta \quad \text{for all } s, t \in [0, 1], 1 \leq i \leq N \text{ with } s, t \in [\sigma_i, \sigma_{i+1}) \\ &\text{and for all } s, t \in [0, \sigma_1) \text{ or } s, t \in [\sigma_N, 1], \end{aligned}$$

and second, $\sum_{t \in N_f \setminus N'_f} |\Delta f(t)|^q < \varepsilon$. For the remainder of the proof we consider $t = 1$. Let $\pi \in \Pi(1)$ with $\text{mesh}(\pi) < \min_{0 \leq i \leq N-1} |\sigma_{i+1} - \sigma_i|$ (where $\sigma_0 := 0$) and, if $\sigma_N < 1$, additionally $\text{mesh}(\pi) < 1 - \sigma_N$. For $\tau = \tau_i \in \pi$ with $1 \leq i \leq |\pi| - 1$ let $\tau^* = \tau_{i+1}$ be its successor. For $1 \leq i \leq N - 1$, let $\tau_i^- \in \pi$ be the largest element strictly smaller than σ_i and $\tau_i^+ = (\tau_i^-)^*$. Then, we have

$$\begin{aligned} &\left| \sum_{i=0}^{|\pi|-1} |f(\tau_{i+1}) - f(\tau_i)|^q - [f]_1^{(q)} \right| \\ &\leq \left| \sum_{i=1}^N |f(\tau_i^+) - f(\tau_i^-)|^q - |\Delta f(\sigma_i)|^q \right| \end{aligned} \quad (5.7)$$

$$+ \sum_{s \in N_f \setminus N'_f} |\Delta f(s)|^q + \sum_{i=1}^N \sum_{\substack{\tau \in \pi \cap [\sigma_i, \sigma_{i+1}) \\ \text{with } \tau^* \in (\sigma_i, \sigma_{i+1}]}} |f(\tau^*) - f(\tau)|^q \quad (5.8)$$

By definition, the first summand in (5.8) does not exceed ε . Moreover,

$$\sum_{i=1}^N \sum_{\substack{\tau \in \pi \cap [\sigma_i, \sigma_{i+1}) \\ \text{with } \tau^* \in (\sigma_i, \sigma_{i+1}]}} |f(\tau^*) - f(\tau)|^q \leq \delta^{q-p} \sum_{i=1}^N \sum_{\substack{\tau \in \pi \cap [\sigma_i, \sigma_{i+1}) \\ \text{with } \tau^* \in (\sigma_i, \sigma_{i+1}]}} |f(\tau^*) - f(\tau)|^p \leq \delta^{q-p} V_p(f).$$

To treat the term in (5.7), note that, for all $x, y \in \mathbb{R}$, we have the elementary inequality

$$||x + y|^q - |x|^q| \leq |y|^q + 2^q(|x|^{q-1}|y| + |x||y|^{q-1}).$$

Applying this inequality to the i -th summand of (5.7) where $x = \Delta f(\sigma_i)$ and $y = f(\tau_i^+) - f(\sigma_i) + f(\sigma_i) - f(\tau_i^-)$, the i -th summand is bounded from above by

$$\begin{aligned} &|\delta_i^+ + \delta_i^-|^q + 2^q(|\Delta f(\sigma_i)|^{q-1}|\delta_i^+ + \delta_i^-| + |\Delta f(\sigma_i)||\delta_i^+ + \delta_i^-|^{q-1}) \\ &\leq 2^q(|\delta_i^+|^q + |\delta_i^-|^q) + 2^q(|\Delta f(\sigma_i)|^{q-1}(|\delta_i^+| + |\delta_i^-|)) \\ &\quad + 4^q|\Delta f(\sigma_i)|(|\delta_i^+|^{q-1} + |\delta_i^-|^{q-1}), \end{aligned}$$

where we have set $\delta_i^+ = f(\tau_i^+) - f(\sigma_i)$ and $\delta_i^- = f(\sigma_i) - f(\tau_i^-)$. It is now straightforward to show that the sum (over $1 \leq i \leq N$) of the last display is bounded from above by $C\delta^\alpha$ for some $0 < \alpha = \alpha(p, q) < 1$ and $C = C(p, q) > 0$. This finishes the proof of (4.5) as δ and ε were chosen arbitrarily. Exemplarily, we pick one of the terms. We have

$$\begin{aligned} \sum_{i=1}^N |\Delta f(\sigma_i)|^{q-1} |\delta_i^+| &\leq \left(\sum_{i=1}^N |\Delta f(\sigma_i)|^q \right)^{1-1/q} \left(\sum_{i=1}^N |\delta_i^+|^q \right)^{1/q} \\ &\leq \delta^{1-p/q} \max_{t \in N_f} |\Delta f(t)|^{(q-p)(1-1/q)} M^{1-1/q} V_p^{1/q}(f). \end{aligned}$$

The regularity of $t \mapsto [f]_t^{(q)}$ and the characterization of its jumps follow immediately.

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